Oscillating Universe like exact solution of the R^2 theory of gravity

[1]Christian Corda and [2]Francesco Fucilla

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[1,2]Institute for Theoretical Physics and Advanced Mathematics Einstein-Galilei, Via Santa Gonda 14, 59100 Prato, Italy and [1,2]Telesio-Galilei Academy of Science, Piazza Indipendenza 7 CH – 6500 Bellinzona Svizzera

[1]Dipartimento di Scienze, Istituto Universitario di Ricerca Scientifica "Santa Rita", Centro di Scienze Naturali, Via di Galceti, 74, 59100 Prato

[1]International Institute for Applicable Mathematics & Information Sciences (IIAMIS), B.M. Birla Science Centre, Adarsh Nagar, Hyderabad - 500 463, India

E-mail addresses: [1]cordac.galilei@gmail.com, [2]anthony.fucilla@btinternet.com

Abstract

In the earlier work Gen. Rel. Grav. 40, 2201-2212 (2008) an oscillating Universe in the linearized R^2 theory of gravity has been analyzed. In that paper the linearized Ricci curvature generated a high energy term which was identified as the Dark Energy of the Universe. The model was consistent with the Hubble Law and the cosmological redshift.

After carefully reviewing previous results, in this work such a model is improved. We show that the line element of the linearized R^2 theory is solution of the exact theory too. The results of earlier linearized approach are therefore consistent with the new model analyzed in the exact R^2 theory. The high energy associated to the Ricci curvature enables the coupling constant of the R^2 term in the gravitational action to be very small with respect to the linear term R. Accordingly, the deviation from standard General Relativity results very weak and the theory can pass the Solar System tests. A quantitative analysis on this critical point is realized too. Other observations, like the anomalous acceleration of the Pioneer and the Dark Matter in the galaxy, are consistent with the model.

The basic approach of this paper is that, in the exact R^2 gravity, a *wave-solution* of a Klein-Gordon equation for the Ricci curvature could solve some problems of the *Dark Universe*. The Ricci curvature represents a *cosmological wave-packet* with a wavelength longer than the Hubble radius. With this assumption the wave-packet is *frozen* with respect to the cosmological observations. Consequently, the theory results to be in the form of standard General Relativity *"embedded"* in an *effective scalar gravity*. General Relativity results dominant a small scales, while the *effective* scalar field, which is exactly the Ricci curvature, oscillates and results dominant at longer scales.

Finally, a cosmological solution to the Einstein-Vlasov System is discussed. This solution shows reasonable results which are within the standard bounds predicted by the cosmological observations.

The model is also consistent with a recent result which shows that the introduction of a non-linear electrodynamics Lagrangian in the framework of R^2 gravity permits to remove the Initial Singularity of the Universe and to obtain a bouncing with a power-law inflation where the Ricci curvature works like an inflaton field.

An important point is that, at the present time, a unique Extended Theory of Gravity which is consistent with all the astronomical observations has not been found. The results of this paper suggest that the cosmological wave-packet could be a potential candidate.

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1 Introduction

Although Einstein's General Relativity [1] achieved great success (see for example the opinion of Landau who says that General Relativity is, together with Quantum Field Theory, the best scientific theory of all [2]) and withstood many experimental tests, it also displayed many shortcomings and flaws which today make theoreticians question whether it is the definitive theory of gravity, see the reviews [3, 4] and references within. As distinct from other field theories, like the electromagnetic theory, General Relativity is very difficult to quantize. This fact rules out the possibility of treating gravitation like other quantum theories, and precludes the unification of gravity with other interactions. At the present time, it is not possible to realize a consistent Quantum Gravity Theory which leads to the unification of gravitation with the other forces. From an historical point of view, Einstein believed that, in the path to unification of theories, quantum mechanics had to be subjected to a more general deterministic theory, which he called Generalized Theory of Gravitation, but he did not obtain the final equations of such a theory (see for example the biography of Einstein in [5]). At present, this point of view is partially retrieved by some theorists, starting from the Nobel Laureate G. 't Hooft [6].

However, one has to recall that, during the last 30 years, a strong, critical discussion about both General Relativity and Quantum Mechanics has been undertaken by theoreticians in the scientific community. The first motivation for this historical discussion arises from the fact that one of the most important goals of Modern Physics is to obtain a theory which could, in principle, show the fundamental interactions as different forms of the same *symmetry* [3, 4]. Considering this point of view, today one observes and tests the results of one or more breaks of symmetry. In this way, it is possible to say that we live in an *unsymmetrical* world. In the last 60 years, the dominant idea has been that a fundamental description of physical interactions arises from Quantum Field Theory. In this tapestry, different states of a physical system are represented by vectors in a Hilbert space defined in a spacetime, while physical fields are represented by operators (i.e. linear transformations) on such a Hilbert space. The greatest problem is that such a Quantum Mechanical framework is not consistent with gravitation, because this particular field, i.e the metric $h_{\mu\nu}$, describes both the dynamical aspects of gravity and the spacetime background. In other words, one says that the quantization of dynamical degrees of freedom of the gravitational field is meant to give a quantum-mechanical description of the spacetime. This is an unequalled problem in the context of Quantum Field Theories, because the other theories are founded on a fixed spacetime background, which is treated like a classical continuum.

Thus, at the present time, an absolute Quantum Gravity Theory, which implies a total unification of various interactions has not been obtained. In addition, General Relativity assumes a classical description of the matter which is totally inappropriate at subatomic scales, which are the scales of the relic Universe [7, 8].

In the unification approaches, from an initial point of view, one assumes that the observed material fields arise from superstructures like Higgs bosons or superstrings which, undergoing phase transitions, generate actual particles. From another point of view, it is assumed that geometry (for example the Ricci curvature scalar R) interacts with material quantum fields generating back-reactions which modify the gravitational action adding interaction terms (examples are high-order terms in the Ricci scalar and/or in the Ricci tensor and non minimal coupling between matter and gravity, see below). Various unification approaches have been suggested, but without palpable observational evidence in a laboratory environment on Earth. Instead, in cosmology, some observational evidences could be achieved with a perturbation approach [8]. Starting from these considerations, one can define *Extended Theories of Gravity* those semi-classical theories where the Lagrangian is modified, in respect of the standard Einstein-Hilbert gravitational Lagrangian, adding high-order terms in the curvature invariants (terms like R^2, $R^{\alpha\beta}R_{\alpha\beta}$, $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$, $R\Box R$, $R\Box^k R$) or terms with scalar fields non-minimally coupled to geometry (terms like $\phi^2 R$), see [3, 4] and references within. In general, one has to emphasize that terms like those are present in all the approaches to the problem of unification between gravity and other interactions. Additionally, from a cosmological point of view, such modifications of General Relativity generate inflationary frameworks which are very important as they solve many problems of the Standard Universe Model [7, 8, 9].

In the general context of cosmological evidence, there are also other considerations which suggest an extension of General Relativity. As a matter of fact, the accelerated expansion of the Universe, which is observed today, implies that cosmological dynamics is dominated by the so called Dark Energy, which gives a large negative pressure. This is the standard picture, in which this new ingredient is considered as a source on the right-hand side of the field equations. It should be some form of un-clustered, non-zero vacuum energy which, together with the clustered Dark Matter, drives the global dynamics. This is the so called "concordance model" (ΛCDM) which gives, in agreement with the CMBR, LSS and SNeIa data, a good picture of the observed Universe today, but presents several shortcomings such as the well known "coincidence" and "cosmological constant" problems [10]. An alternative approach is changing the left-hand side of the field equations, to see if the observed cosmic dynamics can be achieved by extending General Relativity, see [3, 4] and references within. In this different context, it is not required to find candidates for Dark Energy and Dark Matter, that, till now, have not been found; only the "observed" ingredients, which are curvature and baryon matter, have to be taken into account. Considering this point of view, one can think that gravity is different at various scales and there is room for alternative theories. In principle, the most popular Dark Energy and Dark Matter models can be achieved considering $f(R)$ theories of gravity, where R is the Ricci curvature [3, 4]. In this picture, the sensitive detectors for gravitational waves (GWs), like bars and interferometers, whose data analysis recently started [11], could, in principle, be important. In fact, a consistent GW astronomy will be the definitive test for General Relativity or, alternatively, a strong endorsement for Extended Theories of Gravity [12].

In [13], an oscillating Universe has been discussed in the linearized R^2 theory of gravity. The R^2 theory was originally proposed in [9] with the aim of obtaining the cosmological inflation. Such a theory is the simplest among the $f(R)$ theories of gravity and the cosmological redshift and the Hubble law, are consistent with the model in [13].

After carefully reviewing previous results, in this work such a model is improved. The new analysis shows that the line element of the linearized R^2 theory is a solution of the exact theory too. The results of the previous linearized approach in [13] are therefore consistent with the new model analyzed in the exact R^2 theory. The high energy associated to the Ricci scalar curvature enables the coupling constant of the R^2 term in the gravitational action to be very small with respect to the linear term R. Then, the deviation from standard General Relativity results very weak and the theory can pass the Solar System tests. A quantitative analysis of this important point is realized too. Other observations, like the anomalous acceleration of the Pioneer and the Dark Matter in the galaxy, are consistent with the model.

The basic approach of this paper is that, in the exact R^2 gravity, a *wave-solution* of a Klein-Gordon equation for the Ricci scalar could solve some problems of the *Dark Universe*. The Ricci scalar is considered a *cosmological wave-packet* with a wavelength which is longer than the Hubble radius. This assumption implies that the wave-packet is *frozen* with respect to the cosmological observations. Consequently, the theory results to be in the form of standard

General Relativity *"embedded"* in an *effective scalar gravity.* General Relativity results dominant a small scales, while the *effective* scalar field, which is exactly the Ricci scalar, by following the idea in [9], oscillates and results dominant at longer scales.

Finally, a cosmological solution to the Einstein-Vlasov System is discussed. This solution shows reasonable results which are within the standard bounds predicted by the cosmological observations.

The model is also consistent with the recent result in [29], which shows that the introduction of a non-linear electrodynamics Lagrangian in the framework of R^2 gravity permits to remove the Initial Singularity of the Universe and to obtain a bouncing with a power-law inflation where the Ricci curvature scalar works like an inflaton field.

An important point is that, at the present time, a unique Extended Theory of Gravity which is consistent with all the astronomical observations has not been found [3, 4]. The results of this paper suggest that the cosmological wave-packet could be a potential candidate.

2 A review of preview results in the linearized theory

2.1 The linearized field equations

Let us start from the action [9, 13]

$$S = \int d^4x\sqrt{-g}(R + bR^2) + \mathcal{L}_m. \tag{1}$$

Equation (1) is a different choice with respect to the well known canonical one of General Relativity, the Einstein - Hilbert action [2], which is

$$S = \int d^4x\sqrt{-g}R + \mathcal{L}_m. \tag{2}$$

As the gravitational Lagrangian is non-linear in the curvature invariants, the Einstein field equations have an order higher than second [3, 4, 9, 13]. For this reason such theories are called higher-order gravitational theories. This is exactly the case of the action (1). The model arising from the action (1) is well consistent with the temperature anisotropies observed in CMB and thus it can be a viable alternative to the scalar field models of inflation [3, 4].

Before starting the analysis notice that in this paper natural units $8\pi G = 1$, $c = 1$ and $\hbar = 1$ are used, while the sign conventions for the line element, which generate the sign conventions for the Riemann/Ricci tensors, are

(+,-,-,-). Greek indices run from 0 to 3.

By varying the action (1) with respect to $g_{\mu\nu}$ the field equations are obtained [9, 13]

$$G_{\mu\nu}+b\{2R[R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R]+$$
$$-2R_{;\mu;\nu}+2g_{\mu\nu}\Box R\}=T_{\mu\nu}^{(m)}. \tag{3}$$

The trace of such field equations gives a Klein - Gordon equation for the Ricci curvature

$$\Box R=E^2(R+T), \tag{4}$$

where $\Box$ is the d'Alembertian operator. The energy E is introduced for dimensional motivations: $E^2\equiv\frac{1}{6b}$ [9, 13].

In the above equations $T_{\mu\nu}^{(m)}$ is the ordinary stress-energy tensor of the matter. Clearly, General Relativity is obtained for $b=0$.

We re-analyze the linearized theory in vacuum ($T_{\mu\nu}^{(m)}=0$) with a little perturbation of the background, which is assumed given by a Minkowskian background. The computation follows [13] step by step, but in a way which emphasizes the role of the (linearized) Ricci scalar.

Let us write [13]

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}. \tag{5}$$

By considering the first order in $h_{\mu\nu}$ and by calling $\widetilde{R}_{\mu\nu\rho\sigma}$, $\widetilde{R}_{\mu\nu}$ and $\widetilde{R}$ the linearized quantity which correspond to $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ and R, the linearized field equations are

$$\widetilde{R}_{\mu\nu}-\frac{\widetilde{R}}{2}\eta_{\mu\nu}=-\partial_\mu\partial_\nu a\widetilde{R}+\eta_{\mu\nu}\Box a\widetilde{R}$$
$$\Box\widetilde{R}=E^2\widetilde{R}. \tag{6}$$

$\widetilde{R}_{\mu\nu\rho\sigma}$ and eqs. (6) are invariants for gauge transformations [13]

$$h_{\mu\nu}\to h'_{\mu\nu}=h_{\mu\nu}-\partial_{(\mu}\epsilon_{\nu)}$$
$$\widetilde{R}\to\widetilde{R}'=\widetilde{R}. \tag{7}$$

If one defines

$$\bar{h}_{\mu\nu}\equiv h_{\mu\nu}-\frac{h}{2}\eta_{\mu\nu}+\eta_{\mu\nu}b\widetilde{R}, \tag{8}$$

the transform for the parameter ϵ^μ

$$\Box\epsilon_\nu=\partial^\mu\bar{h}_{\mu\nu}, \tag{9}$$

permits to choose a gauge parallel to the Lorenz one of the electromagnetic theory

$$\partial^\mu\bar{h}_{\mu\nu}=0. \tag{10}$$

In this gauge the field equations read like

$$\Box \bar{h}_{\mu\nu} = 0 \tag{11}$$

$$\Box b\widetilde{R} = E^2 b\widetilde{R}. \tag{12}$$

Solutions of eqs. (11) and (12) are plan waves [13]

$$\bar{h}_{\mu\nu} = A_{\mu\nu}(\overrightarrow{p})\exp(ip^{\alpha}x_{\alpha}) + c.c. \tag{13}$$

$$b\widetilde{R} = a(\overrightarrow{p})\exp(iq^{\alpha}x_{\alpha}) + c.c. \tag{14}$$

where

$$\begin{array}{ll} p^{\alpha} \equiv (\omega, \overrightarrow{p}) & \omega = p \equiv |\overrightarrow{p}| \\ & \\ q^{\alpha} \equiv (\omega_E, \overrightarrow{p}) & \omega_E = \sqrt{E^2 + p^2}, \end{array} \tag{15}$$

and a in eq. (14) is a real number.

Eqs. (11) and (13) are the equation and the solution for the tensor waves exactly like in General Relativity [14], while eqs. (12) and (14) are respectively the equation and the solution for the third mode.

The dispersion law for the modes of $\widetilde{R}$ is not linear. The velocity of every "ordinary" (i.e. which is present in standard General Relativity too) mode $\bar{h}_{\mu\nu}$ is the light speed c, but the dispersion law (the second of eq. (15)) for the modes of $\widetilde{R}$ is the one of a *wave-packet* [13]. The group-velocity of a wave-packet centered in $\overrightarrow{p}$ is

$$\overrightarrow{v_G} = \frac{\overrightarrow{p}}{\omega}. \tag{16}$$

From the second of eqs. (15) and eq. (16) one gets

$$v_G = \frac{\sqrt{\omega^2 - E^2}}{\omega}, \tag{17}$$

which can be written as

$$E = \sqrt{(1 - v_G^2)}\omega. \tag{18}$$

The analysis remains in the Lorenz gauge by choosing transformations of the type $\Box\epsilon_{\nu} = 0$. This gauge gives a condition of transverse effect for the ordinary tensor part of the field: $k^{\mu}A_{\mu\nu} = 0$. But it does not give the transverse effect for the total field $h_{\mu\nu}$. From eq. (8) we obtain

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu} + \eta_{\mu\nu}b\widetilde{R}. \tag{19}$$

At this point in standard General Relativity [14] one applies the condition

$$\Box\epsilon^{\mu} = 0$$
$$\partial_{\mu}\epsilon^{\mu} = -\frac{\bar{h}}{2} + b\widetilde{R}. \tag{20}$$

This condition gives the total transverse effect of the field. But, in the present case, this is impossible [13]. In fact, by applying the d'Alembertian operator to the second of eqs. (20) and by using the field equations (11) and (12) we get

$$\Box\epsilon^{\mu} = E^2 b\widetilde{R}, \tag{21}$$

which is in contrast with the first of eqs. (20). This critical point is consistent with claiming that it does not exist any linear relation between the tensor field $\bar{h}_{\mu\nu}$ and the energetic scalar field $\widetilde{R}$ [13]. A gauge in which $h_{\mu\nu}$ is purely spatial cannot be chosen (i.e. we cannot choose $h_{\mu 0} = 0$, see eq. (19)).

But the traceless condition to the pure tensor field $\bar{h}_{\mu\nu}$ can be used [13]

$$\Box\epsilon^{\mu} = 0$$
$$\partial_{\mu}\epsilon^{\mu} = -\frac{\bar{h}}{2}. \tag{22}$$

These equations imply

$$\partial^{\mu}\bar{h}_{\mu\nu} = 0. \tag{23}$$

To enable the conditions $\partial_{\mu}\bar{h}^{\mu\nu}$ and $\bar{h} = 0$, transformations like

$$\partial_{\mu}\Box\epsilon^{\mu} = 0$$
$$\partial_{\mu}\epsilon^{\mu} = 0 \tag{24}$$

can be used. By taking $\overrightarrow{p}$ in the z direction, a gauge in which only A_{11}, A_{22}, and $A_{12} = A_{21}$ are different to zero can be chosen. The condition $\bar{h} = 0$ gives $A_{11} = -A_{22}$. Putting these equations in eq. (19) we obtain

$$h_{\mu\nu}(t,z) = A^{+}(t-z)e_{\mu\nu}^{(+)} + A^{\times}(t-z)e_{\mu\nu}^{(\times)} + b\widetilde{R}(t - v_G z)\eta_{\mu\nu}. \tag{25}$$

The term $A^{+}(t-z)e_{\mu\nu}^{(+)} + A^{\times}(t-z)e_{\mu\nu}^{(\times)}$ describes the two standard tensor polarizations of gravitational waves which are present in General Relativity [14], while the term $b\widetilde{R}(t, -v_G z)\eta_{\mu\nu}$ is the term due to the high energy term in the R^2 theory [13].

In other words, the Ricci scalar results a third polarization for gravitational waves which is not present in standard General Relativity. This third mode associates an intrinsic mass-energy E, and therefore an intrinsic curvature, to the spacetime (see equation (12)) [13].

2.2 The oscillating Universe

Following [13] we assume that, at cosmological scales, the third mode becomes dominant (i.e. $A^+,\ A^- \ll b\widetilde{R}(t,z)$). In the model the "*curvature energy*" is the Dark Energy of the Universe which enables an average density of $\simeq 10^{-29} g/cm^3$ [15].

Eq. (25) can be rewritten as

$$h_{\mu\nu}(t,z) = b\widetilde{R}(t,z)\eta_{\mu\nu} \tag{26}$$

and the correspondent line element is the conformally flat one [13]

$$ds^2 = [1 + b\widetilde{R}(t,z)](dt^2 - dz^2 - dx^2 - dy^2). \tag{27}$$

Defining

$$a^2 \equiv 1 + b\widetilde{R}(t,z), \tag{28}$$

the line element (27) results similar to the cosmological Friedmann-Robertson-Walker (FRW) line element of the standard homogeneous, isotropic and flat Universe [2, 7, 8, 10, 13, 14]

$$ds^2 = [a^2(t,z)](dt^2 - dz^2 - dx^2 - dy^2). \tag{29}$$

Strictly speaking, this metric does not describe an homogeneous and isotropic universe. In fact $\widetilde{R}$ is a function of z too. Thus, we have to further assume $\partial_z\widetilde{R} = 0$, which removes the z-dependence [13, 28]. We will use this constraint in some computations in the following. Thus, the line element becomes

$$ds^2 = a^2(t)(dt^2 - dz^2 - dx^2 - dy^2). \tag{30}$$

Readers could be shocked that we claim to arrive at the metric describing the universe as a perturbation about the Minkowski spacetime. But the key point here is that in Subsection 3.1 we will show that the line element (30) is also solution of the exact theory. In other words, the linearized process that we developed above is only a mathematical tool to obtain the line element (30), while physical consistence will be given in Subsection 3.1 by showing that the line element (30) is solution of the exact theory too. This important point must be keep in mind in the following discussion.

Considering the wave-solution (14) we can write

$$a \simeq 1 + \frac{1}{2}b\widetilde{R}(t). \tag{31}$$

This equation shows that the scale factor of the Universe oscillates near the (normalized) unity.

Before starting the analysis, let us recall that observations today agree with homogeneity and isotropy. In fact, over the past century, a standard cosmological model has emerged. With relatively few parameters, the model describes the evolution of the Universe and astronomical observations on scales ranging from a few to thousands of Megaparsecs. In this model the Universe is spatially flat, homogeneous and isotropic on large scales, composed of radiation, ordinary matter (electrons, protons, neutrons and neutrinos), non-baryonic Cold Dark Matter, and Dark Energy. The galaxies and the large-scale structures grew gravitationally from tiny, nearly scale-invariant adiabatic Gaussian fluctuations [2, 7, 8, 10, 13, 14]. The WMAP data offer a demanding quantitative test of this model [15].

In other words, the Universe is seen like a dynamic and thermodynamic system. The test masses (i.e. the "particles") are the galaxies which are stellar systems with a number of the order of $10^9 - 10^{11}$ stars. The galaxies are located in clusters and super clusters, and observations show that, on cosmological scales, their distribution is uniform. These assumption can be summarized in the Cosmological Principle: *the Universe is homogeneous everywhere and isotropic around every point* [14]. The Cosmological Principle simplifies the analysis of the large scale structure. It implies that the proper distances between any two galaxies is given by an universal scale factor which is the same for any couple of galaxies [2, 7, 8, 10, 13, 14].

Now, the analysis can start.

Cosmological observations are usually carried on Earth and, in any case, within the Solar System. In the linearized theory, the coordinate system in which the space-time is locally flat has to be used and the distance between any two points (the galaxies) is given simply by the difference in their coordinates in the sense of Newtonian physics [14]. This frame is the proper reference frame of a local observer, which we assume to be located within the Solar System. In this frame gravitational signals manifest them-self by exerting tidal forces on the test masses. In other words, we assume that the space-time within the Solar System is locally flat with respect to the global curvature of the Universe which is described by the line element (27) [13].

By using the proper reference frame of a local observer the time coordinate x_0 is the proper time of the observer O and the spatial axes are centered in O. In the special case of zero acceleration and zero rotation the spatial coordinates x_j are the proper distances along the axes and the frame of the local observer reduces to a local Lorentz frame [14]. The line element is

$$ds^2 = (dx^0)^2 - \delta dx^i dx^j - O(|x^j|^2) dx^\alpha dx^\beta. \tag{32}$$

The effect of the gravitational force on test masses is described by the equation

$$\ddot{x}^i = -\widetilde{R}^i_{0k0} x^k, \tag{33}$$

which is the equation for geodesic deviation in this frame [14]. $\widetilde{R}^i_{0k0}$ is the linearized Riemann tensor [14].

To study the effect of the *cosmological wave-packet* on the galaxies, $\widetilde{R}^i_{0k0}$ has to be computed in the proper reference frame of the Solar System. But, because the linearized Riemann tensor $\widetilde{R}_{\mu\nu\rho\sigma}$ is invariant under gauge transformations [14], it can be directly computed from eq. (26).

From [13, 14] we get

$$\widetilde{R}_{\mu\nu\rho\sigma}=\frac{1}{2}\{\partial_\mu\partial_\beta h_{\alpha\nu}+\partial_\nu\partial_\alpha h_{\mu\beta}-\partial_\alpha\partial_\beta h_{\mu\nu}-\partial_\mu\partial_\nu h_{\alpha\beta}\}, \tag{34}$$

that, in the case of eq. (26), gives

$$\widetilde{R}^\alpha_{0\gamma 0}=\frac{b}{2}\{\partial^\alpha\partial_0\widetilde{R}\eta_{0\gamma}+\partial_0\partial_\gamma\widetilde{R}\delta^\alpha_0-\partial^\alpha\partial_\gamma\widetilde{R}\eta_{00}-\partial_0\partial_0\widetilde{R}\delta^\alpha_\gamma\}. \tag{35}$$

The different elements are (only the non zero ones will be written down explicitly) [13]

$$\partial^\beta\partial_0\widetilde{R}\eta_{0\gamma}=\left\{\begin{array}{ccc}\partial_t^2\widetilde{R} & for & \alpha=\gamma=0\\ \\ -\partial_z\partial_t\widetilde{R} & for & \alpha=3;\gamma=0\end{array}\right\} \tag{36}$$

$$\partial_0\partial_\gamma\widetilde{R}\delta^\alpha_0=\left\{\begin{array}{ccc}\partial_t^2\widetilde{R} & for & \alpha=\gamma=0\\ \\ \partial_t\partial_z\widetilde{R} & for & \alpha=0;\gamma=3\end{array}\right\} \tag{37}$$

$$-\partial^\alpha\partial_\gamma\widetilde{R}\eta_{00}=\partial^\alpha\partial_\gamma\widetilde{R}=\left\{\begin{array}{ccc}-\partial_t^2\widetilde{R} & for & \alpha=\gamma=0\\ \\ \partial_z^2\widetilde{R} & for & \alpha=\gamma=3\\ \\ -\partial_t\partial_z\widetilde{R} & for & \alpha=0;\gamma=3\\ \\ \partial_z\partial_t\widetilde{R} & for & \alpha=3;\gamma=0\end{array}\right\} \tag{38}$$

$$-\partial_0\partial_0\widetilde{R}\delta^\alpha_\gamma=\ -\partial_t^2\widetilde{R}\quad for\quad \alpha=\gamma\ . \tag{39}$$

By inserting these results in eq. (35) we obtain [13]

$$\begin{array}{c}\widetilde{R}^1_{010}=-\frac{b}{2}\ddot{\widetilde{R}}\\ \\ \widetilde{R}^2_{010}=-\frac{b}{2}\ddot{\widetilde{R}}\\ \\ \widetilde{R}^3_{030}=\frac{b}{2}(\partial_z^2\widetilde{R}-\partial_t^2\widetilde{R}).\end{array} \tag{40}$$

The constraint on the homogeneity and isotropy implies $\partial_z\widetilde{R}=0$, therefore eqs. (40) become

$$\begin{array}{c}\widetilde{R}^1_{010}=-\frac{b}{2}\ddot{\widetilde{R}}\\ \\ \widetilde{R}^2_{020}=-\frac{b}{2}\ddot{\widetilde{R}}\\ \\ \widetilde{R}^3_{030}=-\frac{b}{2}\ddot{\widetilde{R}}.\end{array} \tag{41}$$

Eqs. (41) show that the oscillations of the Universe are the same in any direction.

In fact, using eq. (33), we obtain

$$\ddot{x} = \frac{b}{2}\ddot{\widetilde{R}}x, \tag{42}$$

$$\ddot{y} = \frac{b}{2}\ddot{\widetilde{R}}y \tag{43}$$

and

$$\ddot{z} = \frac{b}{2}\ddot{\widetilde{R}}z. \tag{44}$$

These are three perfectly symmetric oscillations [13].

2.3 Cosmological observations

2.3.1 The Hubble law

The observations of E. Hubble in 1929 had been the first historical proof of the expansion of the Universe [2, 7, 8, 10, 13, 14]. The Hubble law states that galaxies which are at a distance D drift away from Earth with a velocity [2, 7, 8, 10, 13, 14]

$$v = H_0 D. \tag{45}$$

The today's Hubble expansion rate is [15]

$$H_0 = h_{100}\frac{100Km}{sec \times Mpc} = 3.2 \times 10^{-18}\frac{h_{100}}{sec}. \tag{46}$$

A dimensionless factor h_{100} is included, which now is just a convenience (in the past it came from an uncertainty in the value of H_0). From the WMAP data it is $h_{100} = 0.72 \pm 0.05$ [15].

Following [13] we call f the frequency of the *"cosmological" gravitational wave-packet* that is represented by $b\widetilde{R}$ in eq. (27). We also assume that $f \ll H_0$ (i.e. the gravitational wave is *"frozen"* with respect to the cosmological observations) [13]. The observations of Hubble and the more recent ones show that the oscillating Universe has to be in the expansion phase.

The assumption of homogeneity and isotropy implies that only the radial coordinate can be taken into account.

By using spherical coordinates equations (42), (43) and (43) give an equation for the distance D

$$\ddot{D} = \frac{b}{2}\ddot{\widetilde{R}}D. \tag{47}$$

Equivalently, one can say that there is a gravitational potential [14]

$$V(\overrightarrow{D}, t) = -\frac{d}{4}\ddot{\widetilde{R}}(t)D^2, \tag{48}$$

which generates the tidal forces and that the motion of the test masses is governed by the Newtonian equation [14]

$$\ddot{\vec{r}} = -\bigtriangledown V. \tag{49}$$

The solution of eq. (47) can be found by using the perturbation method [13, 14], obtaining

$$D = D_0 + \frac{b}{2} D_0 \widetilde{R}(t) = (1 + \frac{b}{2}\widetilde{R}) D_0 = a(t) D_0 \tag{50}$$

By deriving this equation with respect to the time we get

$$\dot{D} = D_0 \dot{a} \tag{51}$$

Thus, the Hubble law is obtained

$$\frac{\dot{D}}{D} = H_0, \tag{52}$$

where

$$H_0 = (\frac{\dot{a}}{a})_0. \tag{53}$$

2.3.2 The cosmological redshift

Following [13], the conformally flat line element (29) can be rewritten in spherical coordinates

$$ds^2 = \left[1 + b\widetilde{R}(t)\right]\left[dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \tag{54}$$

The assumption of homogeneity and isotropy permits to neglect the angular coordinates in eq. (54). Then, the condition of null geodesic on the radial coordinate gives

$$dt^2 = dr^2. \tag{55}$$

By using eq. (55) the coordinate velocity of the photon in the gauge (54) is equal to the speed of light. In fact, if we use the coordinates (54), t represents a time coordinate instead of a proper time. The rate $d\tau$ of the proper time (distance) is related to the rate dt of the time coordinate from [2]

$$d\tau^2 = g_{00} dt^2. \tag{56}$$

Eq. (54) gives $g_{00} = (1 + bR)$, and using eq. (55) we obtain

$$d\tau^2 = (1 + b\widetilde{R}))dr^2 \tag{57}$$

and

$$d\tau = \pm[(1 + b\widetilde{R})]^{\frac{1}{2}} dr \simeq \pm[(1 + \frac{b}{2}\widetilde{R})]dr. \tag{58}$$

Assuming that photons are traveling by the galaxy to Earth, we choose the negative sign [13].

Eq. (58) can be integrated obtaining

$$\int_{\tau_1}^{\tau_0} \frac{d\tau}{1+\frac{b}{2}\widetilde{R}(t)} = \int_{r_g}^{0} dr = r_g, \tag{59}$$

where τ_1 and τ_0 are the instants of emission (in the galaxy) and reception (on Earth) of the photon respectively. If the light is emitted with a delay $\triangle\tau_1$ it arrives on Earth with a delay $\triangle\tau_0$. Then

$$\int_{\tau_1}^{\tau_0} \frac{d\tau}{1+\frac{b}{2}\widetilde{R}(t)} = \int_{\tau_1+\triangle\tau_1}^{\tau_0+\triangle\tau_0} \frac{d\tau}{1+\frac{b}{2}\widetilde{R}(t)} = r_g. \tag{60}$$

The radial coordinate r_g is *comoving* (i.e. constant in the gauge (54)). In fact, the assumption of homogeneity and isotropy implies $\partial_z R = 0$, which removes the z dependence in the line element (29). The only dependence in the line element (54) is the t dependence in the scale factor $a = 1 + \frac{b}{2}R(t)$. From equation (60) we get [13]

$$\begin{gathered}\int_{\tau_1}^{\tau_0} \frac{d\tau}{1+\frac{b}{2}R(t)} = \int_{\tau_1}^{\tau_0} \frac{d\tau}{1+\frac{b}{2}R(t)} + \\ + \int_{\tau_1}^{\tau_0+\triangle\tau_0} \frac{d\tau}{1+\frac{b}{2}R(t)} - \int_{\tau_0}^{\tau_1+\triangle\tau_1} \frac{d\tau}{1+\frac{b}{2}R(t)},\end{gathered} \tag{61}$$

which gives

$$\int_{\tau_1}^{\tau_0+\triangle\tau_0} \frac{d\tau}{1+\frac{b}{2}R(t)} = \int_{\tau_0}^{\tau_1+\triangle\tau_1} \frac{d\tau}{1+\frac{b}{2}R(t)}. \tag{62}$$

This equation can be simplified, obtaining

$$\int_{0}^{\triangle\tau_0} \frac{d\tau}{1+\frac{b}{2}R(t)} = \int_{0}^{\triangle\tau_1} \frac{d\tau}{1+\frac{b}{2}R(t)}, \tag{63}$$

and

$$\frac{\triangle\tau_0}{1+\frac{b}{2}R(t_0)} = \frac{\triangle\tau_1}{1+\frac{b}{2}R(t_1)}. \tag{64}$$

Then

$$\frac{\triangle\tau_1}{\triangle\tau_0} = \frac{1+\frac{b}{2}R(t_1)}{1+\frac{b}{2}R(t_0)} = \frac{a(t_1)}{a(t_0)}. \tag{65}$$

Frequencies are inversely proportional to times, thus

$$\frac{f_0}{f_1} = \frac{\triangle\tau_1}{\triangle\tau_0} = \frac{1+\frac{b}{2}R(t_1)}{1+\frac{b}{2}R(t_0)} = \frac{a(t_1)}{a(t_0)}. \tag{66}$$

The definition of the *redshift parameter*

$$z \equiv \frac{f_1 - f_0}{f_0} = \frac{\triangle\tau_0 - \triangle\tau_1}{\triangle\tau_1}, \tag{67}$$

together with eq. (65) gives

$$z = \frac{a(t_0)}{a(t_1)} - 1, \tag{68}$$

which is well known in the literature [2, 7, 8, 10, 13, 14].

Thus, the model looks consistent with the Hubble law and the cosmological redshift.

3 New results

3.1 The line element like solution of the exact R^2 theory

In this subsection we show that the line element (27), which is solution of the linearized field equations (6), is solution of the exact field equations in vacuum

$$\begin{array}{c} G_{\mu\nu} + b\{2R[R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R] + \\ -2R_{;\mu;\nu} + 2g_{\mu\nu}\Box R\} = 0 \end{array} \tag{69}$$

too.

The trace of eq. (69) is

$$\Box R = E^2 R. \tag{70}$$

By inserting the line element (27) in eq. (69), one gets four equations

$$\begin{array}{c} (b\dot{\widetilde{R}})^2 + \frac{4\pi}{3}\frac{(1+b\widetilde{R})^3}{2b\widetilde{R}+1}\left[2b^2\widetilde{R}^3 + b\widetilde{R}^2 - b\widetilde{R} + 12b(b\dot{\widetilde{R}})^2\right] = 0 \\ \\ \frac{b^2}{v_G^2}(\widetilde{R}')^2 + \frac{4\pi}{3}\frac{(1+b\widetilde{R})^3}{2b\widetilde{R}+1}\left[2b^2\widetilde{R}^3 + b\widetilde{R}^2 - b\widetilde{R} + 12b(b\dot{\widetilde{R}})^2\right] = 0, \\ \\ b\ddot{\widetilde{R}} = \frac{(b\dot{\widetilde{R}})^2}{2+2b\widetilde{R}} \\ \\ \frac{b}{v_G^2}\widetilde{R}'' = \frac{(b\widetilde{R}')^2}{v_G^2(2+2b\widetilde{R})} \end{array} \tag{71}$$

where $\dot{\widetilde{R}} \equiv \frac{\partial \widetilde{R}}{\partial t}$ and $\widetilde{R}' \equiv \frac{\partial \widetilde{R}}{\partial z}$.

Eq. (14) implies $\frac{\partial \widetilde{R}}{\partial t} = -v_G \frac{\partial \widetilde{R}}{\partial z}$. Then, one obtains only two independent equations in the system (71), i.e.

$$\begin{array}{c} (b\dot{\widetilde{R}})^2 + \frac{4\pi}{3}\frac{(1+b\widetilde{R})^3}{2b\widetilde{R}+1}\left[2b^2\widetilde{R}^3 + b\widetilde{R}^2 - b\widetilde{R} + 12b(b\dot{\widetilde{R}})^2\right] \\ and \\ b\ddot{\widetilde{R}} = \frac{(b\dot{\widetilde{R}})^2}{2+2b\widetilde{R}}. \end{array} \tag{72}$$

A direct substitution of the wave-packet (14) in eqs. (72) shows that such equations are satisfied.

The result is not totally surprising for two motivations. First, the trace of the linearized field equation in vacuum, i.e. the second of equations (6), is equal to the trace of the exact field equations (70). Second, it is well known from various papers in the literature that gravitational waves are exact solutions of the "full" field equations in standard General Relativity too [16, 17, 18, 19].

Subsequent to this analysis we argue that if one uses the line element (27) and/or the wave-packet (14) the linearized theory coincides with the exact theory. Thus, in the following analysis we can substitute the linearized Ricci curvature $\widetilde{R}$ with the ordinary one R. The line element (27) is rewritten as

$$ds^2 = [1 + bR(t,z)](dt^2 - dz^2 - dx^2 - dy^2). \tag{73}$$

3.2 Viability of the model

The R^2 theory is the simplest among the class of viable models with R^m terms. Such models may lead to the (cosmological constant or quintessence) acceleration of the universe as well as an early time inflation [3, 4]. Moreover, they seem to pass the Solar System tests, i.e. they have an acceptable Newtonian limit, no instabilities and no Brans-Dicke problem (decoupling of scalar) in scalar-tensor version.

By assuming a value of $\simeq 10^{-29} g/cm^3$ [15] for the average density of the Dark Energy of the Universe, we get $a \simeq 10^{-46} cm^4$ in natural units. This enables the constant coupling of the R^2 term in the gravitational action to be infinitesimal with respect to the linear term R. The variation from standard General Relativity is very weak and the theory can pass the Solar System tests. Regarding this important issue, it is important to provide citations of antecedent work illustrating this and explicitly show that the bounds there agree with the preferred values of the Dark Energy. The key point is that, as the effective scalar field arising from curvature is very energetic, the constant coupling of the the R^2 non linear term $\rightarrow 0$. In this case, the Ricci curvature, which is an extra dynamical quantity in the metric formalism, must have a range longer than the size of the Solar System. An important work is [20], where it is shown that this is correct if the effective length of the scalar field l is much shorter than the value of $0.2mm$. In such a case, the presence of this effective scalar is hidden from Solar System and terrestrial experiments. The value of the Dark Energy that we are assuming here guarantees the condition $l \ll 0.2mm$. Another important test concerns the deflection of light by the Sun. This effect was studied in R^2 gravity by calculating the Feynman amplitudes for photon scattering. To linearized order, this deflection is the same as in standard General Relativity [21].

For a sake of completeness, in the following the viability of the model is discussed by using a different analysis.

The condition $f \ll H_0$ guarantees that the Ricci curvature remains *frozen*, i.e. constant, in respect to the scale of the Solar System. Thus, one puts

$$bR = K = constant. \tag{74}$$

Let us search variations from standard General Relativity within the Solar System by considering a spherically symmetric Schwarzschild-like metric generated by the solar mass $M_\odot$ with the corrections that are generated by the Ricci curvature [22]

$$ds^2 = -\exp(-\lambda r)dt^2 + \exp(\lambda r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (75)$$

Following [22], the Ricci scalar is given by

$$R = \exp(-\lambda r)(\lambda'' - \lambda'^2 + \frac{4\lambda'}{r} - \frac{2}{r^2}) + \frac{2}{r^2}, \qquad (76)$$

where $'$ represents derivation in respect to r.

By inserting the condition (74) in eq. (76) we get [22]

$$\lambda(r) = -\ln(\frac{\alpha}{r} - \frac{\beta}{r^2} - \frac{K}{12b}r^2). \qquad (77)$$

We choose $\alpha = -2M$, $\beta = 0$ in analogy with the standard Schwarzschild metric, then [2, 22]

$$ds^2 = -(1 - \frac{2M}{r} - \frac{K}{12b}r^2)dt^2 + (1 - \frac{2M}{r} + \frac{K}{12b}r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (78)$$

This metric is similar to the Schwarzschild–de Sitter space in the standard Einstein-Hilbert action with cosmological constant [2, 22]. $R = \frac{K}{b} = 4\Lambda$ plays the role of the cosmological constant. The calculation of the perihelion of Mercury in standard General Relativity with cosmological constant has been shown in [23]. The result was a constraint $\Lambda < 10^{-55}cm^{-2}$. This implies a constraint on the *frozen* Ricci scalar: $R < 10^{-55}cm^{-2}$. In this way, the deviation of the spacetime (73) from flatness within the Solar System is infinitesimal at the present cosmological Era: $bR = K < 10^{-101}$.

The geometrical interpretation of the above analysis is that the theory results to be in the form of standard General Relativity *"embedded"* in an *effective scalar gravity*. General Relativity results dominant a small scales like the scale of the Solar System. The *effective* scalar field, which is exactly the Ricci curvature, oscillates and results dominant at longer scales.

3.3 Astrophysical observations: Pioneer anomaly and Dark Matter within the galaxy

In the linearized theory computations are simper than computations in the exact theory. Thus, we continue to use the linearized theory in this subsection by keeping in mind that the linearized theory and the exact theory are equivalent for the wave-packet R.

Considering the scales of the galaxy and of the Solar System we have to remove the assumption of homogeneity and isotropy. Eqs. (40) become

$$\begin{aligned}\widetilde{R}^1_{010} &= -\tfrac{b}{2}\ddot{\widetilde{R}}\\ \widetilde{R}^2_{010} &= -\tfrac{b}{2}\ddot{\widetilde{R}}\\ \widetilde{R}^3_{030} &= \tfrac{b}{2}\Box\widetilde{R}.\end{aligned} \tag{79}$$

Inserting the field equation (12) in the third of eqs. (79) we get

$$\widetilde{R}^3_{030} = \frac{1}{2}E^2\widetilde{R}. \tag{80}$$

Eq. (33) gives three oscillations

$$\ddot{x} = \frac{b}{2}\ddot{\widetilde{R}}(t,z)x, \tag{81}$$

$$\ddot{y} = \frac{b}{2}\ddot{\widetilde{R}}(t,z)y \tag{82}$$

and

$$\ddot{z} = -\frac{b}{2}E^2\widetilde{R}(t,z)z. \tag{83}$$

The resultant of these three oscillations represents an extra acceleration $\overrightarrow{a}_e$. Again, one assumes $f \ll H_0$, which guarantees that the Ricci curvature R remains *frozen*, i.e. constant, in respect to the galactic scale. This assumption implies that the extra acceleration depends only by the position.

By adding the standard Newtonian acceleration one obtains

$$\overrightarrow{a}_{tot} = \overrightarrow{a}_n + \overrightarrow{a}_e \tag{84}$$

where the total acceleration $\overrightarrow{a}_{tot}$ is given by the ordinary Newtonian acceleration $\overrightarrow{a}_n$ plus the extra acceleration $\overrightarrow{a}_e$.

As the extra acceleration depends by the position, only phenomenology can help in its identification.

In a galactic context it is natural to identify a_e with $a_0 \simeq 10^{-10}m/s$, which is the acceleration used in the theoretical context of Modified Newtonian Dynamics to achieve Dark Matter into galaxies [24].

From another point of view, in the Solar System, if the anomaly in Pioneer acceleration [25] is not generated by systematic effects, but a real effect is present, one can in principle put

$$a_e = a_{Pi} \simeq 8.5 \times 10^{-10}m/s^2. \tag{85}$$

Thus, the proposed approach allows for a unified explanation of the two effects.

3.4 Cosmology of the Einstein-Vlasov system

In this subsection, following the analysis in [28], the Einstein-Vlasov system [26, 27, 28] is analyzed for the homogeneous isotropic spacetime generated by the line element (73). For a Universe consisting of galaxies with negligible baryon mass (massless particles in the Einstein-Vlasov system), the equation for the scale factor is solved analytically. Considering galaxies like massless particles represents a good approximation [28]. In fact, astronomical observations show that Dark Matter and Dark Energy, that we discussed like pure effects of curvature in previous analyses, represent the 95% of the mass-energy of the Universe [10, 15]. The remaining 5% of baryon mass is neglected in the following analysis.

The condition that the particles in the spacetime make up an ensemble with no collisions is satisfied if the particle density is a solution of the Vlasov equation [26, 27, 28]

$$\partial_t f + \frac{p^a}{p^0}\partial_{x^a} f - \Gamma^a_{\mu\nu}\frac{p^\mu p^\nu}{p^0}\partial_{p^a} f = 0, \tag{86}$$

where $\Gamma^\alpha_{\mu\nu}$ are the Christoffel coefficients, f is the particle density and p^0 is given by $p^a (a = 1, 2, 3)$ according to the relation [26, 27, 28]

$$g_{\mu\nu}p^\mu p^\nu = -1. \tag{87}$$

Eq. (87) implies that the four momentum p^μ lies on the mass-shell of the spacetime [26, 27, 28].

We recall that, in general, the Einstein-Vlasov system is given by [26, 27, 28]

$$\begin{gathered}\partial_t f + v \cdot \bigtriangledown_x f - \bigtriangledown_x U \cdot \bigtriangledown_v f = 0 \\ \bigtriangledown \cdot U = 4\pi\rho \\ \rho(t, x) = \int dv f(t, x, v),\end{gathered} \tag{88}$$

where t denotes the time and x and v the position and the velocity of the galaxies. The function $U = U(t, x)$ is the average Newtonian potential generated by the galaxies. This system represents the non-relativistic kinetic model for an ensemble of particles with no collisions interacting through gravitational forces which they generate collectively [26, 27, 28].

Thus, such a system can be used for a description of the motion of the galaxies in the Universe if galaxies are considered as point-like particles, and the relativistic effects are negligible[26, 27, 28]. In this approach, the function $f(t, x, v)$ in the Einstein-Vlasov system (88) gives the density on phase space of the galaxies within the Universe.

By using the classical transformation from conformal time to synchronous time [14, 28]

$$dt \to \frac{dt}{\sqrt{1 + bR(t, z)}} \tag{89}$$

the line element (73), in spherical coordinates, becomes

$$ds^2 = dt^2 - [1 + bR(t,z)](dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)). \tag{90}$$

The metric tensor has the form [28]

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ & \\ 0 & \gamma_{mn} \end{pmatrix}, \tag{91}$$

where $\gamma_{mn} = 1 + bR(t,z)$.

Following [2, 28] we define $\chi_{mn} \equiv \frac{\partial}{\partial t}\gamma_{mn}$. The Einstein field equations in the synchronous frame are:

$$R_0^0 = -\frac{1}{2}\frac{\partial}{\partial t}\chi_a^a - \frac{1}{4}\chi_a^b\chi_b^a = (T_0^0 - \frac{1}{2}T) \tag{92}$$

$$R_a^0 = \frac{1}{2}(\chi_{a;b}^b - \chi_{b;a}^a) = T_a^0 \tag{93}$$

$$R_a^b = -P_a^b - \frac{1}{2\sqrt{\gamma}}\frac{\partial}{\partial t}(\sqrt{\gamma}\chi_a^b) = T_a^b - \frac{1}{2}\delta_a^b T, \tag{94}$$

where P_a^b is the Ricci tensor in 3 dimensions [2, 28].

On the other hand, the Einstein field equations in the Einstein-Vlasov system are [26, 27, 28]

$$G_{\mu\nu} = \frac{2}{\sqrt{-g}}\int f(t,x,p)p_\mu p_\nu \delta(p^2 + m^2)d^4p. \tag{95}$$

We can split the function $f(t,x,p)$ into a couple of equations for $f_+(t,x,p)$ and $f_-(t,x,p)$ which are constructed by reducing $f(t,x,p)$ respectively on the "upper" half and on the "lower" half of the mass shell [28]. Eq. (86) becomes [28]

$$\partial_t f_\pm = -\frac{1}{p_\pm^0}\left(\gamma^{mn}p_n\frac{\partial}{\partial x_m} - \frac{1}{2}\frac{\partial\gamma^{nr}}{\partial x_m}p_n p_r\frac{\partial}{\partial p_m}\right)f_\pm. \tag{96}$$

Eq. (96) can be interpreted in Hamiltonian terms [28]:

$$p_\pm^0\partial_t f_\pm = \{H, f_\pm\}, \tag{97}$$

where the Hamiltonian function is

$$H \equiv \frac{1}{2}\gamma^{mn}p_m p_n. \tag{98}$$

One can calculate the components of energy-momentum tensor $T_{\mu\nu}$ in the approximation which considers galaxies like massless particles ($m = 0$ in eq. (95)) [28]

$$T_{00} = \frac{1}{(\sqrt{1+bR(t,z)})^3 r^2 \sin\theta} \int \frac{f_+ + f_-}{\sqrt{1+bR(t,z)}} \sqrt{\frac{p_1^2+p_2^2}{r^2} + \frac{p_3^2}{r^2\sin\theta}} d^3p \quad (99)$$

$$T_{mn} = \frac{1}{(\sqrt{1+bR(t,z)})^3 r^2 \sin\theta} \int \sqrt{1+bR(t,z)} \frac{(f_+ + f_-)}{\sqrt{\frac{p_1^2+p_2^2}{r^2} + \frac{p_3^2}{r^2\sin\theta}}} p_m p_n d^3p \quad (100)$$

$$T_{0m} = \frac{1}{(\sqrt{1+bR(t,z)})^3 r^2 \sin^2\theta} \int (f_+ - f_-) p_m d^3p. \quad (101)$$

The Einstein field equations (92), (93) and (94) give two independent dynamic equations which can be written down in terms of the scale factor $a = \sqrt{1+bR(t,z)}$ [28]:

$$\dot{a}^2 = -1 + \frac{1}{3a} \int (f_+(s) + f_-(s)) \frac{s}{a} d^3s \quad (102)$$

$$\ddot{a} = -\frac{2}{a} - 2\frac{\dot{a}^2}{a} + \frac{1}{a^2} \int (f_+(s) + f_-(s)) d^3s, \quad (103)$$

where [28]

$$s \equiv p_1^2 + \frac{p_2^2}{r^2} + \frac{p_3^2}{r^2\sin^2\theta}. \quad (104)$$

By introducing the dimensionless variables $\underline{r}$ and $\underline{t}$ we put [28]

$$\begin{aligned} a &= a_0 \underline{r} \\ t &= a_0 \underline{t} \\ \dot{\underline{r}} &= \tfrac{d\underline{r}}{d\underline{t}} \\ j &= \tfrac{1}{3} a_0^2 \rho_0. \end{aligned} \quad (105)$$

Eq. (102) becomes [28]

$$\begin{aligned} \dot{\underline{r}}^2 &= -1 + \tfrac{j}{\underline{r}^2} \\ \underline{r}_0 &= 1. \end{aligned} \quad (106)$$

The solution of the system (106) is [28]

$$\underline{r}(\underline{t}) = \sqrt{j - (\underline{t} - \sqrt{j-1})^2} \quad (107)$$

if $j \geq 1$. Returning to the $(t,\, a)$ variables we get:

$$a(t) = a_0\sqrt{\frac{a_0^2\rho_0}{3} - (\frac{t}{a_0} - \sqrt{\frac{a_0^2\rho_0}{3}} - 1)^2}. \qquad (108)$$

The today's observed Hubble radius and the today's observed density of the Universe are respectively [15] $a_0 \gtrsim 10^{28}cm$ and $\rho_0 \approx 10^{-57}cm^{-2}$. Therefore $j \approx 1$.

By inserting these data in eq. (108) we obtain a singularity at a time $t_s \approx -10^{10}seconds$ and a value for the today's theoretical Hubble constant $H_0 = \frac{\dot{a}_0}{a_0} \approx 10^{-29}cm^{-1}$.

As follows from the above analysis, even under the assumption to neglect the baryon mass of the galaxies the results look reasonable. They are of the same order of magnitude of the standard cosmological model [15, 28]. The singularity can, in principle, be removed, by following [29], if one introduces a non-linear electrodynamics Lagrangian in the framework, which permits also to obtain a bouncing with a power-law inflation where the Ricci scalar curvature works like an inflaton field, see [29] for details. Clearly, the bouncing is in agreement with the oscillating Universe.

We hope in further analyses which could insert the baryon mass too and realize a better *fine-tuning* of the model with the cosmological observations.

4 Conclusions

We discussed an exact solution of the R^2 theory of gravity. This solution represents the Ricci scalar curvature like a *cosmological wave-packet* with a wavelength longer than the Hubble radius. The physical interpretation is that the theory results to be in the form of standard General Relativity *"embedded"* in an *effective scalar gravity*. General Relativity results dominant a small scales like the scale of the Solar System while the *effective* scalar field, which is is exactly the Ricci scalar curvature, oscillates and results dominant at longer scales.

The proposed solution is viable and the analysis is consistent with various astronomical observations like the Hubble Law, the cosmological redshift, the anomalous acceleration of the Pioneer and the Dark Matter in the galaxy. The final treatment in terms of the Einstein-Vlasov System shows reasonable results which are within the standard bounds predicted by the cosmological observations.

The model is also consistent with the recent result in [29], which shows that the introduction of a non-linear electrodynamics Lagrangian in the framework of R^2 gravity permits to remove the Initial Singularity of the Universe and to obtain a bouncing with a power-law inflation where the Ricci scalar works like an inflaton field.

An important point is that, at the present time, a unique Extended Theory of Gravity which is consistent with *all* the astronomical observations has not been found [3, 4]. The results of this paper suggest that the *cosmological wave-packet* could be a potential candidate. We hope in further analyses to realize a better *fine-tuning* of the model with the cosmological observations.

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