On the Hodge dual of the first Bianchi identity

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One of the basic equations of differential geometry is the 1st Bianchi identity

(1) $D \wedge T^a = R^a{}_b \wedge q^b$.

In their web note [1] the authors attempt to dualize that identity by claiming the Hodge dual of eq.(1) to be

(1~) $D \wedge T^{a\sim} = R^a{}_b{}^\sim \wedge q^b$

as an up to now unknown further *basic equation of differential geometry.*

To prove that new equation (1~) is transformed to the tensorial equation

(2) $D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}$

which is considered to be *equivalent* to the dualized 1st Bianchi identity (1~). Therefore the tensorial equation (2) should be valid for arbitrary sample manifolds of Riemannian differential geometry. We'll check this assertion (2) by an elementary example below. The reader will find a much deeper treatment of the topic by W.A. Rodrigues Jr. in [4].

1. The unit-2-sphere S^2 in R^3

Using some basic information from S.M. Carroll [3] we have the metric

(1.1) $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 = dx^{1\,2} + \sin^2 x^1\, dx^{2\,2}$

with the metric tensors

(1.2) $(g_{\mu\nu}) = \mathrm{diag}(\ 1, \sin^2 x^1)$, $(g^{\mu\nu}) = \mathrm{diag}(\ 1, {}^1/_{\sin^2 x^1})$.

using the numbering of indices

(1.3) $1 \sim \theta,\ 2 \sim \varphi$

There are only a few non-vanishing Christoffel coefficients

(1.4) $\Gamma^1{}_{22} = -\sin x^1 \cos x^1$, $\Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot x^1$,

while all other Christoffels $\Gamma^\kappa{}_{\mu\nu}$ vanish.

The torsion T^κ is given by

(1.5) $$T^\kappa{}_{\mu\nu} = \Gamma^\kappa{}_{\mu\nu} - \Gamma^\kappa{}_{\nu\mu} = 0 ,$$

i.e. vanishing due to the symmetry of the Christoffels in their lower indices μ,ν.

2. The Riemann tensor of S^2

The Riemann tensor is given by

(2.1) $$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\kappa{}_{\lambda\nu} - \partial_\nu\Gamma^\kappa{}_{\lambda\mu} + \Gamma^\kappa{}_{\mu\rho}\Gamma^\rho{}_{\nu\lambda} - \Gamma^\kappa{}_{\nu\rho}\Gamma^\rho{}_{\mu\lambda}$$

being antisymmetric in μ,ν as is well-known. Therefore we have especially

(2.3) $$R^\kappa{}_{\lambda\mu\nu} = 0 \quad \text{if} \quad \mu = \nu.$$

3. The check

Due to the vanishing of torsion (1.5) the equation (2) to be checked reduces to

(3.1) $$0 = R^\kappa{}_\mu{}^{\mu\nu} = R^\kappa{}_{\mu\alpha\beta}\, g^{\mu\alpha}\, g^{\nu\beta} .$$

Therefore, due to the diagonal form (1.2) of $(g^{\mu\rho})$, we have to check:

(3.2) $$(R^\kappa{}_{11\beta}\, g^{11} + R^\kappa{}_{22\beta}\, g^{22})\, g^{\nu\beta} = (R^\kappa{}_{111}\, g^{11} + R^\kappa{}_{221}\, g^{22})\, g^{\nu 1} + (R^\kappa{}_{112}\, g^{11} + R^\kappa{}_{222}\, g^{22})\, g^{\nu 2} .$$

This is:

for ν=1: $R^\kappa{}_\mu{}^{\mu 1} = (R^\kappa{}_{111}\, g^{11} + R^\kappa{}_{221}\, g^{22})\, g^{11} + 0 = (R^\kappa{}_{111}\, g^{11} + R^\kappa{}_{221}\, g^{22})\, g^{11} = R^\kappa{}_{221}\, g^{22}\, g^{11}$,

for ν=2: $R^\kappa{}_\mu{}^{\mu 2} = 0 + (R^\kappa{}_{112}\, g^{11} + R^\kappa{}_{222}\, g^{22})\, g^{22} = (R^\kappa{}_{112}\, g^{11} + R^\kappa{}_{222}\, g^{22})\, g^{22} = R^\kappa{}_{112}\, g^{11}\, g^{22}$,

i.e. the test reduces to:

(3.3) $$R^\kappa{}_{221} = 0 \,? \qquad \text{and} \qquad R^\kappa{}_{112} = 0 \,?$$

We consider the special case $\kappa = 1$ to obtain:

(3.4) $$R^1{}_{221} = 0 \,? \qquad \text{and} \qquad R^1{}_{112} = 0 \,?$$

The check '$R^1{}_{221} = 0$?' means in detail

(3.5) $$R^1{}_{221} = \underbrace{\partial_2\Gamma^1{}_{21}}_{=0} - \partial_1\Gamma^1{}_{22} + (\underbrace{\Gamma^1{}_{21}}_{=0}\Gamma^1{}_{12} + \Gamma^1{}_{22}\Gamma^2{}_{12}) - (\underbrace{\Gamma^1{}_{11}}_{=0}\Gamma^1{}_{22} + \underbrace{\Gamma^1{}_{12}}_{=0}\Gamma^2{}_{22}) = 0 \,?$$

thus

$$(3.6) \qquad R^1{}_{221} = -\,\partial_1\Gamma^1{}_{22} + \Gamma^1{}_{22}\Gamma^2{}_{12} = -\,\sin^2 x^1 \neq 0\ .$$

Therefore we have obtained a **negative check result**: The test equation (2) is not fulfilled for the unit-2-sphere S^2 which means:

Eq.(2) is invalid in general.

Remark: Another counter example to eq.(2) is given by the Schwarzschild metric in [2]: Sect.1.1.4 gives symmetric Christoffel connection, hence the torsion is zero. However, due to Sect.1.1.12 we have $R^{o}{}_{\mu}{}^{\mu o} \neq 0$, again contradicting eq.(2).

References

[1] M.W. Evans, H. Eckardt, *Violation of the Dual Bianchi Identity by Solutions of the Einstein Field Equation*
Violation of the Dual Bianchi Identity

[2] M.W. Evans, H. Eckardt, *Spherically symmetric metric with perturbation a/r*
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[3] S.M. Carroll, *Lecture Notes on General Relativity*, p.60 f.,
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[4] W.A. Rodrigues Jr., *Differential Forms on Riemannian (Lorentzian) and Riemann-Cartan Structures and Some Applications to Physics*
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