A remark on antisymmetric connections

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A recent web publication [1, Abstract and conclusion (15) ?⇒? (16)] leads to the question whether a linear connection $\Gamma^\rho{}_{\mu\nu}$ of a pseudo-Riemanian manifold *M* can be antisymmetric with respect to *all* coordinate bases, i.e. whether

(1) $\Gamma^\rho{}_{\mu\nu} = -\,\Gamma^\rho{}_{\nu\mu}$ (antisymmetry of connection)

remains valid also under arbitrary local changes

(2) $x^{\mu'} = x^{\mu'}(x^\mu)$

of the coordinate basis.

The answer can easily be given by considering the transformation behaviour of the connection coefficients as reported here from a private communication by W.A. Rodrigues Jr.:

Any coordinate transformation (2) causes a transformation of the connection coefficient $\Gamma^\rho{}_{\mu\nu} \to \Gamma^{\rho'}{}_{\mu'\nu'}$ to be specified here:

Let

(3) $a^{\mu'}{}_\mu := \partial x^{\mu'}/\partial x^\mu$ and $a^\mu{}_{\mu'} := \partial x^\mu/\partial x^{\mu'}$

denote the transformation coefficients of the coordinate transformation (2). Then, as is well known (see introductory textbooks e.g. [2, p.56]), the connection transforms as follows:

(4) $\Gamma^{\rho'}{}_{\mu'\nu'} = \Gamma^\rho{}_{\mu\nu}\, a^{\rho'}{}_\rho\, a^\mu{}_{\mu'}\, a^\nu{}_{\nu'} - a^\mu{}_{\mu'}\, a^\nu{}_{\nu'}\, \partial a^{\rho'}{}_\mu/\partial x^\nu$

The first term on the right hand side does not disturb the symmetry behaviour: If $\Gamma^\rho{}_{\mu\nu}$ is symmetric/antisymmetric in μ,ν then so is $\Gamma^\rho{}_{\mu\nu}\, a^{\rho'}{}_\rho$: in μ',ν' symmetric/antisymmetric respectively. However, the second term is of interest: Due to

(5) $a^\nu{}_{\nu'}\, a^\mu{}_{\mu'}\, \partial a^{\rho'}{}_\nu/\partial x^\mu = a^\mu{}_{\mu'}\, a^\nu{}_{\nu'}\, \partial a^{\rho'}{}_\mu/\partial x^\nu$ (since $\partial a^{\rho'}{}_\nu/\partial x^\mu = \partial^2 x^{\rho'}/\partial x^\mu \partial x^\nu = \partial a^{\rho'}{}_\mu/\partial x^\nu$)

this term is term is always *symmetric* in μ',ν'. So if antisymmetry is wanted then this term does not play with and spoils the wanted antisymmetry in general:

Therefore we have the following result:

A coordinate transformation (2) *preserves symmetry* of the connection in the two lower indices while *antisymmetry is NOT preserved* in general.

Therefore the answer to our introductory question is negative: A connection *antisymmetric in all possible coordinate bases* cannot exist.

References

[1] M.W. Evans, *ON THE SYMMETRY OF THE CONNECTION IN RELATIVITY AND ECE THEORY*,
http://www.aias.us/documents/uft/a122ndpaper.pdf

[2] S.M. Carroll, *Lecture Notes on General Relativity*, Chapter 3

[3] G.W. Bruhn, *Commentary on Evans' web note #122*,
http://www.mathematik.tu-darmstadt.de/~bruhn/onEvansNote122.html

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Links

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Riemannian manifold

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