The Schwarzschild 0riginal Solution and the Droste and Hilbert Solutions of Einstein Equations and Black-Holes. A Resume of a Long Story and the Many Misunderstandings it Produced.

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Abstract

I briefly recall that the original Schwarzschild solution and the Droste and Hilbert solutions (and their maximal extension) to Einstein equations are in fact *different* solutions which define manifolds with very different topologies. Moreover, whereas the original Schwarzschild solution does not imply in the existence of back-holes, the (maximal extension) of the Droste-Hilbert solution defines a manifold (Kruskal space-time) that contains a black-hole. All the above statements are pure mathematical ones and cannot be disputed. They have nothing to do with the possible existence or not of black-holes in our physical world. Keeping this in mind put immediately an end to polemics that one find from time to time in the literature.

1. Schwarzschild [1] looked for a solution of Einstein equations supposing a priori that the space-time manifold where a point mass and the gravitational field it generated live is $M = \mathbb{R} \times \mathbb{R}^3$ where time takes values in $\mathbb{R}$ and $\mathbb{R}^3$ denotes the usual three-dimensional Euclidean space. Indeed, he equipped $\mathbb{R} \times \mathbb{R}^3$ with coordinates (t, x, y, z) explicitly saying that (x, y, z) are *rectangular Cartesian coordinates*. After that, as a second step he introduced usual polar coordinate functions in the game. In so doing, he correctly left out from $\mathbb{R}^3$ the origin $\mathbf{O} = (0, 0, 0)$ where the point mass particle is supposed to be located at any instant of time and start solving Einstein equations in the manifold $\mathbb{R} \times (\mathbb{R}^3 - \mathbf{O}) \approx \mathbb{R} \times (0, \infty) \times \mathbf{S}^2$, introducing standard spherical coordinates $(\boldsymbol{r}, \vartheta, \phi)$. However, since those coordinate functions do not satisfy the Einstein coordinate gauge that Schwarzschild need to use he introduced *spherical coordinates with determinant* 1. After some mathematical tricks, he found as solution of his problem a metric field $\boldsymbol{g}_S$, which has a unique singularity at $\mathbf{O}$ and as such his solution *does not imply in any black-hole*.

2. However, in fact Schwarzschild, in order to determine one of the integration constants of the differential equations he was solving, needed for his calculations to use the *manifold with boundary*[1] $\mathbb{R}\times[0,\infty)\times\boldsymbol{S}^2$ and thus his original mass point supposed to be located for any instant of time at the point $\mathbf{O}\in\mathbb{R}^3$ ended to be represented by the manifold[2] $\{0\}\times\boldsymbol{S}^2$ (something obviously odd that Hilbert elegantly, without criticizing Schwarzschild, observed in a footnote of his paper on the Schwarzschild solution).

3. Before proceeding, and in order to avoid any confusion note that despite the fact that the original manifold postulated as model of space-time by Schwarzschild is $\mathbb{R}\times\mathbb{R}^3$ this does not imply that this manifold or the manifold $\mathbb{R}\times[0,\infty)\times\boldsymbol{S}^2$ equipped with the Levi-Civita connection $\boldsymbol{D}$ of $\boldsymbol{g}_S$ (that solves Einstein equation) is *flat*. In fact, the connection $\boldsymbol{D}$ for Schwarzschild problem is curved, this statement meaning that its Riemann curvature tensor is non null. Please, take always this into account [3]:

"Manifolds do not have curvature, it is the connection imposed on a manifold that may or may not have non null curvature (*and/or non null torsion, non null nonmetricty*). *Some manifolds may be bended surfaces in a Euclidean* (*or pseudo-Euclidean*) *space of appropriate dimension. But to be* ***bended*** (*a property described by the so called shape tensor*) *has in general nothing to do with the fact that a connection defined in the manifold is curved."*

4. Droste [2] and Hilbert [4] found independently another solution of Einstein equations based on different assumptions than the ones used by Schwarzschild[3]. Modern relativists[4] (following Droste and Hilbert) find as solution[5] with *rotational symmetry* of Einstein equations in vacuum a metric field $\boldsymbol{g}_{DH}$ (at least C^2) defined in the manifold $\mathbb{R}\times(0,2m)\cup(2m,\infty)\times\boldsymbol{S}^2$. Relativists say that the "part" $\mathbb{R}\times(0,2m)\times\boldsymbol{S}^2$ where the solution is valid defines a black-hole.

5. It is crucial to have in mind that the *quasi spherical coordinates functions* (r,ϑ,ϕ) used by modern relativists are such that the coordinate function r is not the Schwarzschild spherical coordinate function $\boldsymbol{r}$, i.e.,

$$r\neq\boldsymbol{r}.$$

[1] For the use of the concept of manifold with voundary to present singularities in General Relativity, see, [5, 9, 10, 8]. For some skifull commentaries on the peril of using boundary manifolds in General Relativity without taking due care see [11]

[2] $\{0\}$ denotes the set whose unique element is $0\in[0,\infty)$, i.e., the *boundary* of the semi-line $[0,\infty)$.

[3] In the words of Synge [12]: Schwarzschild imposed spherical symmetry, whereas Droste and Hilbert imposed rotationalsymmetry, a subtle but crucial detail.

[4] See, e.g., [5, 7]

[5] Eventually, it would better to say, as did O'Neill in his book [7] that we start looking for *one* solution of a problem and ended with *two* solutions.

6. Schwarzschild wrote his final formula for $\boldsymbol{g}_S$ using a function $R(\boldsymbol{r})$ which is formally identical to the Droste-Hilbert formula for $\boldsymbol{g}_{DH}$ if $R(\boldsymbol{r})$ is read as the coordinate function r. However, Schwarzschild solution is valid only for $R(\boldsymbol{r}) > 2m$ whereas the Droste-Hilbert solution is valid for any $r \in (0, 2m) \cup (2m, \infty)$.

7. Of course, there is no sense in supposing that space-time has a *disconnected* topology. Thus, under the present *ideology* of finding maximal extension of manifolds equipped with Lorentzian metrics as the true representatives of gravitational fields, relativists maximally extend the solution $\boldsymbol{g}_{DH}$ to a solution $\boldsymbol{g}$ valid in a connected manifold called the *Kruskal* (sometimes, Kruskal-Szekeres) space-time [6, 13]. The total Kruskal manifold which has an exotic topology describes a hypothetical object called the *wormhole.* The final solution $\boldsymbol{g}$ is presented as a function of the coordinate functions $(u, v, \vartheta, \varphi)$ and r which (*keep this in mind*) becomes an implicit function of the coordinate functions (u, v).

8. It is assumed by relativists that a connected "part" of the Kruskal manifold describes a black-hole where $\boldsymbol{g}$ has a real singularity only at the place defined by the function $r(u, v) = 0$.

9. In conclusion, Schwarzschild original solution and the Kruskal extension of the Droste-Hilbert solution define space-times with very *different* topologies, so they are not the same solution of Einstein equations. In the former the topology of the manifold has been fixed a priori, in the latter the topology of the manifold has been fixed a posteriori by the process of maximal extension.

10. There are some published papers that do not properly distinguish these two different solutions[6], moreover, there are some authors stating (explicitly, or in a disguised way) that it is possible to extend the Schwarzschild original solution that was written in terms of the function $R(\boldsymbol{r})$ for the domain $0 \leq R(\boldsymbol{r}) < \infty$, but this idea is, of course, a logical non sequitur, since for Schwarzschild the manifold is fixed a priori. Any appropriate discussion of the mathematical aspect of the back-hole solution of Einstein equations clearly requires a reasonable understanding of differential geometry, and of course, of topology[7]. And it is also important, to advise that everyone that wants to discuss the black-hole issue and did not read the original Schwarzschild paper (or its English version, available at the **arXiv**) must do that in a hurry.

11. Failing to properly understand the different topologies of the two solutions mentioned above (Schwarzschild and the maximal extension of the Droste-Hilbert) is thus making some people (including some that say to be relativist physicists) not to discuss modern relativity theory, but some other things, believing to be the same thing.

12. *The question if black holes exist or not is, of course, not a mathematical*

[6] Besides that there are also some non sequitur mathematical statements in some of those papers.

[7] No more than the what may be found in [5] or [7] .

one[8], *it is a physical question* and I presently believe that they do not exist, leaving this clear in my last book [3], where I argue that it is necessary to construct a theory of the gravitational field where that field is to be regarded as a field in the sense of Faraday (like the electromagnetic field and the weak and strong force fields) "living" in Minkowski space-time[9]. *Thus, that "part" of the maximal extension of the Droste-Hilbert solution of Einstein equations (describing a black hole) probably does not describe anything real in the physical world.*

References

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[8] It is indeed out of question, the fact that Einstein equations (according to the modern interpretation of General Relativity theory) have solutions describing black-holes. Of course, this does not leave everyone happy and many physicists have proposed and are proposing alternative solutions of Einstein equations capable of describing the final stage of super dense stars and which according to them looks more "realistic".

[9] However, I am prepared to change my mind, if empirical facts (stronger than the ones that have appeared until now in the literature) make me do to do so. Moreover, let me say that having dedicated a lot of time to the study of Falaco solitons in the last months I can even envisage the possibility that eventually we live in a space which is the surface of a 3-dimensional brane separating two regions of different densities of a primordial substance (which live in a manifold with at least dimensions besides the usual time dimension) and that particles are topological defects in the basic structure. To learn what is a Falaco soliton, please give a look on the paper (in Portuguese) that my very young student Samuel Wainer (working now for a Ph.D. in Mathematics) wrote last year when he was an undergraduate student in Physics [14]. See also a movie done by Professor Robert Kiehn that teaches how to generate a Falaco soliton at http://www22.pair.com/csdc/car/carfre94.htm and also the many (not so easy to read) papers that he wrote on the subject.

[10] This is an English translation due to S. Antoci and A. Loinger of Schwarzschild original paper published at *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, Phys.-Math.Klasse*, 189-196 (1916).

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[11] An English version is being prepared.