

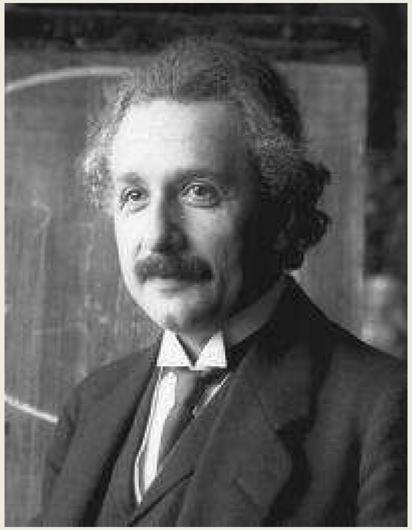


Geometry and the Nature of Gravitation

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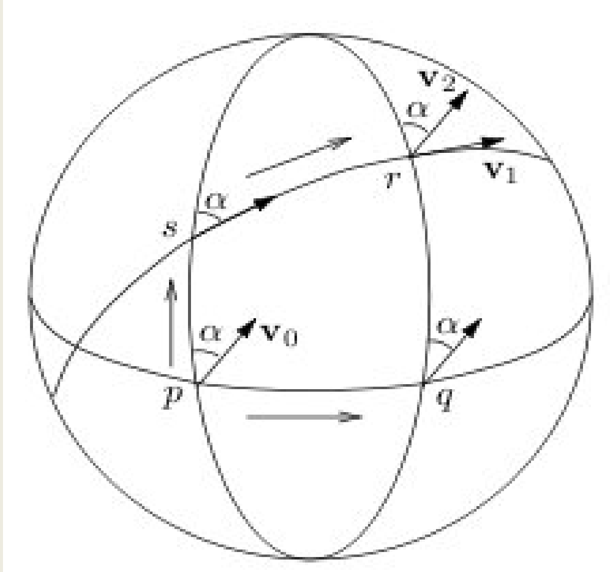
IMECC-UNICAMP

ICCA9-WEIMAR 2011

The Flat and the Curved Punctured Sphere
Do not Confuse Curvature with Bending

Levi Civita Connection D and Nunes Connection ▽



$$(x^1,x^2)=(\vartheta,\varphi),\ 0<\vartheta<\pi,\ 0<\varphi<2\pi,\ \left(\boldsymbol{e}_1=\frac{\partial}{\partial x^1},\ \boldsymbol{e}_2=\frac{1}{\sin x^1}\frac{\partial}{\partial x^2}\right),\ \left(\theta^1=dx^1,\theta^2=\sin x^1 dx^2\right)$$

$$[\boldsymbol{e}_i,\boldsymbol{e}_j]=c^k_{ij}\boldsymbol{e}_k,\ c^2_{12}=-c^2_{21}\cot x^1,\ g=dx^1\otimes dx^1+\sin^2 x^1 dx^2\otimes dx^2.$$

• *Levi - Civita Connection* $D\boldsymbol{g}=0$

$$D_{\partial_\mu}\partial_\nu=\Gamma^\rho_{\mu\nu}\partial_\rho,\ \Gamma^2_{21}=\Gamma^\varphi_{\varphi\vartheta}=\Gamma^2_{12}=\Gamma^\varphi_{\vartheta\varphi}=\cot\vartheta,\ \Gamma^1_{22}=\Gamma^\vartheta_{\varphi\varphi}=-\cos\vartheta\sin\vartheta,$$

$$D_{\boldsymbol{e}_i}\boldsymbol{e}_j=\omega^k_{ij}\boldsymbol{e}_k,\quad \omega^2_{21}=\cot\vartheta,\quad \omega^1_{22}=-\cot\vartheta.$$

$$\mathcal{T}^D\left(\theta^k,\boldsymbol{e}_i,\boldsymbol{e}_j\right)=\theta^k\left(\tau^D\left(\boldsymbol{e}_i,\boldsymbol{e}_j\right)\right)=\theta^k\left(D_{\boldsymbol{e}_i}\boldsymbol{e}_j-D_{\boldsymbol{e}_j}\boldsymbol{e}_i-[\boldsymbol{e}_i,\boldsymbol{e}_j]\right)=0,$$

$$\mathcal{T}^D:=\frac{1}{2}T^k_{ij}\theta^i\wedge\theta^j\otimes\boldsymbol{e}_k=\boldsymbol{\Theta}^k\otimes\boldsymbol{e}_k,\ \boldsymbol{\Theta}^k:=\frac{1}{2}T^k_{ij}\theta^i\wedge\theta^j$$

$$\boldsymbol{R}^D(\boldsymbol{e}_k,\theta^a,\boldsymbol{e}_i,\boldsymbol{e}_j)=\theta^a\left(\left(D_{\boldsymbol{e}_j}D_{\boldsymbol{e}_i}-D_{\boldsymbol{e}_i}D_{\boldsymbol{e}_j}-D_{[\boldsymbol{e}_i,\boldsymbol{e}_j]}\right)\boldsymbol{e}_k\right),$$

$$R_1{}^1{}_{21}=-R_1{}^1{}_{12}=R_1{}^2{}_{12}=-R_1{}^2{}_{21}=-1.$$

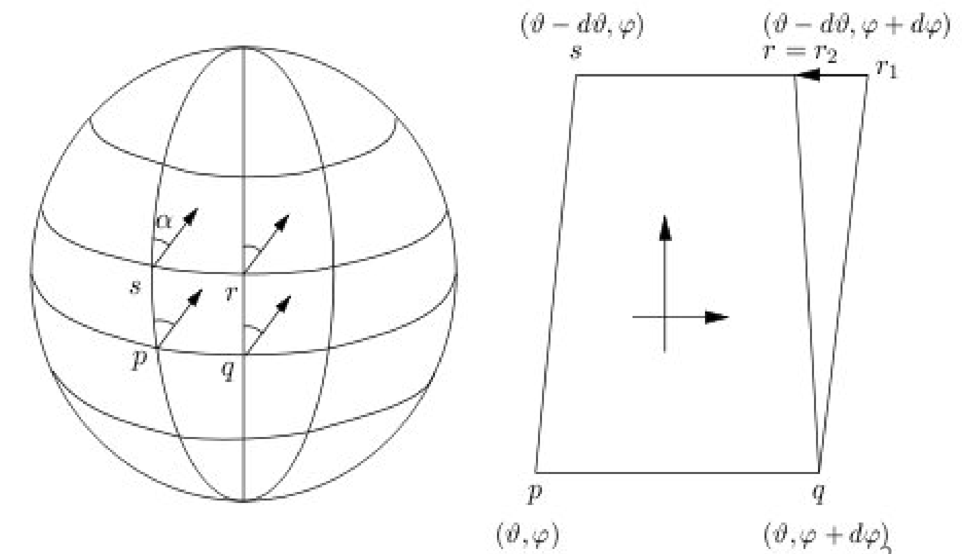
• *Nunes Connection* $\nabla\boldsymbol{g}=0$

$$\nabla_{\boldsymbol{e}_i}\boldsymbol{e}_j=0,$$

$$\boldsymbol{R}^\nabla=0,$$

$$\tau^\nabla\left(\boldsymbol{e}_i,\boldsymbol{e}_j\right)=\nabla_{\boldsymbol{e}_j}\boldsymbol{e}_i-\nabla_{\boldsymbol{e}_i}\boldsymbol{e}_j-[\boldsymbol{e}_i,\boldsymbol{e}_j]=-[\boldsymbol{e}_i,\boldsymbol{e}_j],$$

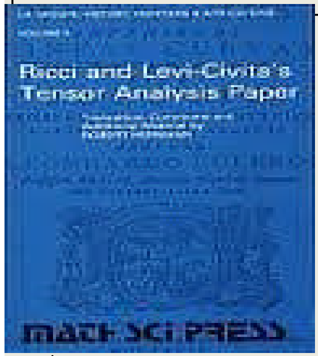
$$T^2_{21}=-T^2_{12}=-\cot\vartheta.$$



(a) (b)

The Gravitational Field in *GR*

• *GR* models a gravitational field generated by a given energy-momentum tensor T by a Lorentzian spacetime $\langle M, D, \boldsymbol{g}, \tau_g, \uparrow\rangle$, where M is a noncompact (locally compact) 4-d Hausdorff manifold, $\boldsymbol{g}$ is a Lorentzian metric field, D is the Levi-Civita connection of $\boldsymbol{g}$, $\tau_g \in \sec \Lambda^4 T^*M$ is a orientation, $\uparrow$ is a time orientation.



• Einstein Equation : $\boxed{Ricci - \frac{1}{2}\boldsymbol{g}R = -\boldsymbol{T} \Leftrightarrow \boldsymbol{\mathcal{R}}^{\mathrm{a}} - \frac{1}{2}R\boldsymbol{\mathfrak{g}}^{\mathrm{a}} = -\boldsymbol{T}^{\mathrm{a}}},$

$Ricci = R_{\mathrm{ab}}\boldsymbol{\mathfrak{g}}^{\mathrm{a}} \otimes \boldsymbol{\mathfrak{g}}^{\mathrm{b}}$, $\boldsymbol{\mathcal{R}}^{\mathrm{a}} = R^{\mathrm{a}}_{\mathrm{b}}\boldsymbol{\mathfrak{g}}^{\mathrm{b}} \in \sec \Lambda^1\, T^*M$ are the *Ricci 1- form fields*, $\quad (\mathcal{E})$

$\boldsymbol{g}(\boldsymbol{e}_{\mathrm{a}}, \boldsymbol{e}_{\mathrm{b}}) = \eta_{\mathrm{ab}}$, $\boldsymbol{\mathfrak{g}}^{\mathrm{a}}(\boldsymbol{e}_{\mathrm{b}}) = \delta^{\mathrm{a}}_{\mathrm{b}}$, $\{\boldsymbol{e}_{\mathrm{a}}\}$ basis of TM, $\{\boldsymbol{\mathfrak{g}}^{\mathrm{a}}\}$ basis of T^*M,
matrix with entries η_{ab} is $diag(1,-1,-1,-1)$.

• Problems with the GR model:

1. Which is the topology of *M*?

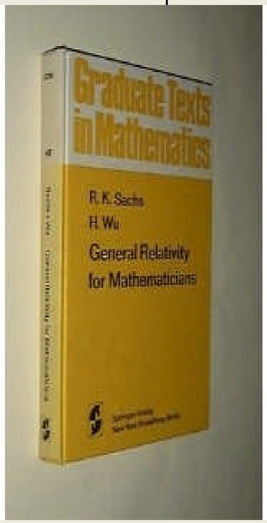
2. There are no conservation laws for energy-momentum and angular momentum in *GR*.

" As mentioned in section 3.8, conservation laws have a great predictive power.

It is a shame to lose the special relativistic total energy conservation law (Section 3.10.2) in general relativity.

Many of the attempts to resurrect it are quite interesting; many are simply garbage."

(Sachs&Wu, General Relativity for Mathematicians, pp. 97-98, Springer 1977)



• Possible solutions:

• (i) represent the gravitational field by a different geometrical model, e.g., e.g.,a *teleparallel spacetime* $\langle M, \nabla, g, \tau_g, \uparrow\rangle$;

• (ii) represent the gravitational field as a field in Faraday sense living in *Minkowski* spacetime $\langle M, \overset{\circ}{D}, \eta, \tau_\eta, \uparrow\rangle$.

Alternative Representation for a Reliable Gravitational Field

- Suppose M is *parallelizable*, i.e.: $\exists$ four *global* vector fields $e_{\mathrm{a}} \in \sec TM$, $\mathbf{a} = 0,1,2,3$, and $\{e_{\mathrm{a}}\}$ is a *basis* for TM.
- **Motivation** is *Geroch theorem*: Necessary and sufficient condition for a 4-dimensional Lorentzian manifold $\langle M, g\rangle$ to admit spinor fields is that the orthonormal frame bundle be trivial. (thus parallelizable)

(Geroch, R. Spinor Structure of Space-Times in General Relativity I, *J. Math. Phys.* **9**, 1739-1744 (1968).)

- Let $\{\mathfrak{g}^{\mathrm{a}}\}$, $\mathfrak{g}^{\mathrm{a}} \in \sec T^{*}M$ be the corresponding *dual basis*, $\mathfrak{g}^{\mathrm{a}}(e_{\mathrm{b}}) = \delta^{\mathrm{a}}_{\mathrm{b}}$. Suppose moreover that not all $\mathfrak{g}^{\mathrm{a}}$ are *closed*, i.e., $d\mathfrak{g}^{\mathrm{a}} \neq 0$, for at least some $\mathbf{a} = 0,1,2,3$.
- $\mathfrak{g}^{0} \wedge \mathfrak{g}^{1} \wedge \mathfrak{g}^{2} \wedge \mathfrak{g}^{3} \in \sec \Lambda^{4} T^{*}M$ defines a (positive) orientation for M.
- Define a *Lorentzian metric* in M by $\boldsymbol{g} = \eta_{\mathrm{ab}} \mathfrak{g}^{\mathrm{a}} \otimes \mathfrak{g}^{\mathrm{b}}$. Define $g = \eta^{\mathrm{ab}} e_{\mathrm{a}} \otimes e_{\mathrm{b}} \in \sec T^{2}_{0} M$.
- e_{0} is a *global timelike vector field* (according to $\boldsymbol{g}$) and $\uparrow$ defines a *time orientation* for M.
  - Since some of the $d\mathfrak{g}^{\mathrm{a}} \neq 0$, we have $[e_{\mathrm{a}}, e_{\mathrm{b}}] = c^{\mathrm{k}}_{\mathrm{ab}} e_{\mathrm{k}}$ and moreover $d\mathfrak{g}^{\mathrm{a}} = -\frac{1}{2} c^{\mathrm{k}}_{\mathrm{ab}} \mathfrak{g}^{\mathrm{a}} \wedge \mathfrak{g}^{\mathrm{b}}$.
- Next introduce *two different* metric compatible connections in M, namely D, the Levi-Civita connection of $\boldsymbol{g}$, and ∇, a *teleparallel* connection.

$$D_{e_{\mathrm{a}}} e_{\mathrm{b}} = \omega^{\mathrm{k}}_{\mathrm{ab}} e_{\mathrm{k}}, \quad D_{e_{\mathrm{a}}} \mathfrak{g}^{\mathrm{b}} = -\omega^{\mathrm{b}}_{\mathrm{ak}} \mathfrak{g}^{\mathrm{k}}, \qquad \nabla_{e_{\mathrm{a}}} e_{\mathrm{b}} = 0, \quad \nabla_{e_{\mathrm{a}}} \mathfrak{g}^{\mathrm{b}} = 0.$$

- $\omega^{\mathrm{k}}_{\mathrm{a}} = \omega^{\mathrm{k}}_{\mathrm{ab}} \mathfrak{g}^{\mathrm{b}}$ are the connection 1-forms of D and $\bar{\omega}^{\mathrm{k}}_{\mathrm{a}} = \bar{\omega}^{\mathrm{k}}_{\mathrm{ab}} \mathfrak{g}^{\mathrm{b}} = 0$ are the connection 1-forms of ∇ in the basis $\{e_{\mathrm{a}}\}$.

►Then we immediately have two possible structures:

- $\langle M, D, g, \tau_g, \uparrow\rangle$, $\boldsymbol{\Theta}^{\mathbf{a}} := d\mathfrak{g}^{\mathbf{a}} + \omega^{\mathbf{a}}_{\mathbf{b}} \wedge \mathfrak{g}^{\mathbf{b}} = 0$, $\quad \mathcal{R}^{\mathbf{a}}_{\mathbf{b}} := d\omega^{\mathbf{a}}_{\mathbf{b}} + \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} \neq 0$;

where $\boldsymbol{\Theta}^{\mathbf{a}} \in \sec \Lambda^2 T^*M$ and $\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \in \sec \Lambda^2 T^*M$ are respectively the torsion and curvature 2-forms of D.

- $\langle M, \nabla, g, \tau_g, \uparrow\rangle$, $\boldsymbol{\mathcal{F}}^{\mathbf{a}} := d\mathfrak{g}^{\mathbf{a}} + \varpi^{\mathbf{a}}_{\mathbf{b}} \wedge \mathfrak{g}^{\mathbf{b}} = d\mathfrak{g}^{\mathbf{a}}$, $\overset{\varpi}{\mathcal{R}}{}^{\mathbf{a}}_{\mathbf{b}} := d\varpi^{\mathbf{a}}_{\mathbf{b}} + \varpi^{\mathbf{a}}_{\mathbf{c}} \wedge \varpi^{\mathbf{c}}_{\mathbf{b}} = 0$;

where $\boldsymbol{\mathcal{F}}^{\mathbf{a}} = d\mathfrak{g}^{\mathbf{a}} \in \sec \Lambda^2 T^*M$ and $\overset{\varpi}{\mathcal{R}}{}^{\mathbf{a}}_{\mathbf{b}} \in \sec \Lambda^2 T^*M$ are respectively the torsion and curvature 2-forms of ∇.

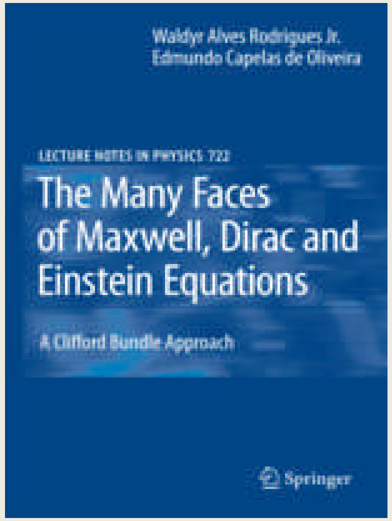
- Now, before proceeding we suppose that:

$$\boxed{\Lambda T^*M = \oplus_{r=0}^{4} \Lambda^r T^*M \hookrightarrow \mathcal{C}\ell(M,g),}$$

where $\mathcal{C}\ell(M,g)$ is the *Clifford bundle of non homogeneous differential forms* and we use the conventions about the scalar product, left and right contractions and the Hodge star operator $\underset{g}{\star}$ and the Hodge coderivative operator $\underset{g}{\delta}$ as in the book:

W. A. Rodrigues Jr. and E. Capelas de Oliveira, *The Many Faces of Maxwell, Dirac and Einstein Equations. A Clifford Bundle Approach,* Springer 2007.

errata at: http://www.ime.unicamp.br/~walrod/errata11012010.pdf



The Potentials $\{\mathfrak{g}^{\mathrm{a}}\}$ as Representatives of the Gravitational Field

Lagrangian Density : $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m$

$\mathcal{L}_g$ is the gravitational Lagrangian

$\mathcal{L}_m$ is the matter Lagrangian

Postulate :

$$\boxed{\mathcal{L}_g = -\frac{1}{2} d\mathfrak{g}^{\mathrm{a}} \wedge \underset{g}{\star} d\mathfrak{g}_{\mathrm{a}} + \frac{1}{2} \underset{g}{\delta} \mathfrak{g}^{\mathrm{a}} \wedge \underset{g}{\star} \underset{g}{\delta} \mathfrak{g}_{\mathrm{a}} + \frac{1}{4} (d\mathfrak{g}^{\mathrm{a}} \wedge \mathfrak{g}_{\mathrm{a}}) \wedge \underset{g}{\star} (d\mathfrak{g}^{\mathrm{b}} \wedge \mathfrak{g}_{\mathrm{b}}).} \quad (\mathcal{L})$$

The form of this Lagrangian is notable, the first term is Yang-Mills like, the second one is a kind of gauge fixing term and the third term is an autointeraction term describing the interaction of the *vorticities* of the potentials.

Before proceeding we observe that this Lagrangian is not invariant under arbitrary point *dependent Lorentz rotations* of the basic cotetrad fields. In fact, if $\mathfrak{g}^{\mathrm{a}} \mapsto \mathfrak{g}'^{\mathrm{a}} = \Lambda^{\mathrm{a}}_{\mathrm{b}} \mathfrak{g}^{\mathrm{b}} = R \mathfrak{g}^{\mathrm{a}} \tilde{R}$, $\forall x \in M$, where $\Lambda^{\mathrm{a}}_{\mathrm{b}}(x) \in L^{\uparrow}_{+}$, the homogeneous and orthochronous Lorentz group and $R(x) \in Spin_{1,3} \subset \mathbb{R}_{1,3}$ we get

$$\boxed{\mathcal{L}'_g - \mathcal{L}_g = \text{exact differential.}}$$

So, the field equations derived from the variational principle results invariant under a change of gauge.

The Field Equations

$$\boxed{d \underset{g}{\star} \mathcal{S}_{\mathrm{d}} + \underset{g}{\star} t_{\mathrm{d}} = - \underset{g}{\star} \mathcal{T}_{\mathrm{d}} ,} \qquad (1)$$

$$\begin{aligned} \underset{g}{\star}\mathcal{S}_{\mathrm{d}} &:= \frac{\partial \boldsymbol{\mathcal{L}}_g}{\partial d\mathfrak{g}^{\mathrm{d}}} = -\mathfrak{g}^{\mathrm{a}} \underset{g}{\star}(d\mathfrak{g}_{\mathrm{a}} \wedge \mathfrak{g}_{\mathrm{d}}) + \frac{1}{2}\mathfrak{g}_{\mathrm{d}} \wedge \underset{g}{\star}(d\mathfrak{g}^{\mathrm{a}} \wedge \mathfrak{g}_{\mathrm{a}}) \\ &= -\underset{g}{\star}\mathcal{F}_{\mathrm{d}} - (\mathfrak{g}_{\mathrm{d}} \underset{g}{\lrcorner} \underset{g}{\star} \mathfrak{g}^{\mathrm{a}}) \wedge \underset{g}{\star} d \underset{g}{\star} \mathfrak{g}_{\mathrm{a}} + \frac{1}{2}\mathfrak{g}_{\mathrm{d}} \wedge \underset{g}{\star}(d\mathfrak{g}^{\mathrm{a}} \wedge \underset{g}{\star}\mathfrak{g}_{\mathrm{a}}), \\ &\qquad \boxed{\mathcal{F}_{\mathrm{d}} = d\mathfrak{g}_{\mathrm{d}}} \end{aligned} \qquad (2)$$

$$\begin{aligned} \underset{g}{\star} t_{\mathrm{d}} &:= \frac{\partial \boldsymbol{\mathcal{L}}_g}{\partial \mathfrak{g}^{\mathrm{d}}} = \mathfrak{g}_{\mathrm{d}} \underset{g}{\lrcorner} \boldsymbol{\mathcal{L}}_g - (\mathfrak{g}_{\mathrm{d}} \underset{g}{\lrcorner} d\mathfrak{g}^{\mathrm{a}}) \wedge \frac{\partial \boldsymbol{\mathcal{L}}_g}{\partial d\mathfrak{g}^{\mathrm{d}}} \\ &= \frac{1}{2}(\mathfrak{g}_{\mathrm{d}} \underset{g}{\lrcorner} d\mathfrak{g}^{\mathrm{a}}) \wedge \underset{g}{\star} d\mathfrak{g}_{\mathrm{a}} - d\mathfrak{g}^{\mathrm{a}} \wedge (\mathfrak{g}_{\mathrm{d}} \underset{g}{\lrcorner} \underset{g}{\star} d\mathfrak{g}^{\mathrm{a}}) \\ &+ \frac{1}{2} d(\mathfrak{g}_{\mathrm{d}} \underset{g}{\lrcorner} \underset{g}{\star} \mathfrak{g}^{\mathrm{a}}) \wedge \underset{g}{\star} d \underset{g}{\star} \mathfrak{g}_{\mathrm{a}} + \frac{1}{2}(\mathfrak{g}_{\mathrm{d}} \underset{g}{\lrcorner} d \underset{g}{\star} \mathfrak{g}^{\mathrm{a}}) \wedge \underset{g}{\star} d \underset{g}{\star} \mathfrak{g}_{\mathrm{a}} + \frac{1}{2} d\mathfrak{g}_{\mathrm{d}} \wedge \underset{g}{\star}(\mathfrak{g}^{\mathrm{a}} \wedge d\mathfrak{g}_{\mathrm{a}}) \\ &- \frac{1}{4}(d\mathfrak{g}^{\mathrm{a}} \wedge \mathfrak{g}_{\mathrm{a}}) \wedge \underset{g}{\star}[\mathfrak{g}_{\mathrm{d}} \underset{g}{\lrcorner} \underset{g}{\star}(d\mathfrak{g}^{\mathrm{c}} \wedge \mathfrak{g}_{\mathrm{c}})] - \frac{1}{4}[\mathfrak{g}_{\mathrm{d}} \underset{g}{\lrcorner}(d\mathfrak{g}^{\mathrm{c}} \wedge \mathfrak{g}_{\mathrm{c}})] \wedge \underset{g}{\star}(d\mathfrak{g}^{\mathrm{a}} \wedge \mathfrak{g}_{\mathrm{a}}), \end{aligned} \qquad (3)$$

$$\boxed{\underset{g}{\star}\mathcal{T}_{\mathrm{d}} = \frac{\partial \boldsymbol{\mathcal{L}}_m}{\partial \mathfrak{g}^{\mathrm{d}}} = -\underset{g}{\star} T_{\mathrm{d}}.} \qquad (4)$$

Maxwell Like Form of the Field Equations

- Recall that $\mathcal{F}^{\mathrm{d}} = d\mathfrak{g}^{\mathrm{d}}$ and define:

$$\boxed{\mathfrak{h}_{\mathrm{d}} := d[(\mathfrak{g}_{\mathrm{d}} \wedge \underset{g}{\star} \mathfrak{g}^{\mathrm{a}}) \wedge \underset{g}{\star} d\mathfrak{g}_{\mathrm{a}} - \frac{1}{2}\mathfrak{g}_{\mathrm{d}} \wedge \underset{g}{\star}(\mathcal{F}^{\mathrm{a}} \wedge \underset{g}{\star} \mathfrak{g}_{\mathrm{a}})],} \quad (5)$$

and a *possible legitimate energy momentum tensor for the gravitational field*

$$\boxed{\mathfrak{t}_{\mathrm{d}} = \mathfrak{h}_{\mathrm{d}} + t_{\mathrm{d}}.} \quad (6)$$

Then, recalling Eq.(1) and the definition of $\underset{g}{\delta}$, we can write the Maxwell like

equations for the gravitational field: $$\boxed{\text{(a) } d\mathcal{F}^{\mathrm{d}} = 0, \text{ (b) } \underset{g}{\delta}\mathcal{F}_{\mathrm{d}} = -(\mathcal{T}_{\mathrm{d}} + \mathfrak{t}_{\mathrm{d}}).} \quad (7)$$

- Compare Eqs.(7a) and (7b) with Maxwell Equations in the Structure $\langle M, \boldsymbol{g}, \tau_g, \uparrow\rangle$.

$A \in \sec \Lambda^1 T^* M$, $J_e \in \sec \Lambda^1 T^* M$, $F = dA \in \sec \Lambda^2 T^* M$,

$$\boxed{dF = 0, \quad \underset{g}{\delta} F = -J_e.} \quad (8)$$

The Many Faces of Maxwell and Einstein Equations

• *Dirac operator* acting on $A_r \in \sec \Lambda^r T^*M \hookrightarrow \mathcal{C}\ell(M,g)$ in $\langle M, D, \boldsymbol{g}, \tau_g, \uparrow \rangle$:

$$\partial := \mathfrak{g}^{\mathrm{a}} D_{e_{\mathrm{a}}},\ \partial = \partial\wedge + \partial \underset{g}{\lrcorner} = d - \underset{g}{\delta};\quad \partial\wedge A_r = dA_r,\ \ \partial \underset{g}{\lrcorner} A_r = -\underset{g}{\delta} A_r. \qquad (9)$$

• *Square of* ∂:

$$\partial^2 A_r = (\partial\wedge\partial)A_r + (\partial\cdot\partial)A_r;\ \partial^2 A_r = -(d\underset{g}{\delta} + d\underset{g}{\delta})A_r,$$

$$(\partial\wedge\partial) \neq -d\underset{g}{\delta},\ (\partial\wedge\partial) \neq -\underset{g}{\delta} d;\ (\partial\cdot\partial) \neq -d\underset{g}{\delta},\ (\partial\cdot\partial) \neq -\underset{g}{\delta} d \qquad (10)$$

$(\partial\wedge\partial)$ is called the Ricci operator and: $(\partial\wedge\partial)\mathfrak{g}^{\mathrm{a}} = \boldsymbol{\mathcal{R}}^{\mathrm{a}} = R^{\mathrm{a}}_{\mathrm{b}}\ \mathfrak{g}^{\mathrm{b}} \in \sec\Lambda^1 T^*M \hookrightarrow \mathcal{C}\ell(M,g)$,

$\square = (\partial\cdot\partial)$ is the covariant D'Alembertian, $\blacklozenge = -(d\underset{g}{\delta} + d\underset{g}{\delta}) = \partial^2$ is called Hodge Laplacian.

• *Dirac operator* acting on $A_r \in \sec \Lambda^r T^*M \hookrightarrow \mathcal{C}\ell(M,g)$ in $\langle M, \nabla, \boldsymbol{g}, \tau_g, \uparrow \rangle$:

$$\boldsymbol{\partial} := \mathfrak{g}^{\mathrm{a}} \nabla_{e_{\mathrm{a}}},\ \boldsymbol{\partial} = \boldsymbol{\partial}\wedge + \boldsymbol{\partial}\underset{g}{\lrcorner};\quad \boldsymbol{\partial}\wedge A_r = dA_r - \boldsymbol{\mathcal{F}}^{\mathrm{a}} \wedge (\mathfrak{g}_{\mathrm{a}} \underset{g}{\lrcorner} A_r),\quad \boldsymbol{\partial}\underset{g}{\lrcorner} A_r = -\underset{g}{\delta} A_r - \boldsymbol{\mathcal{F}}^{\mathrm{a}} \underset{g}{\lrcorner} (\mathfrak{g}_{\mathrm{a}} \wedge A_r). \qquad (10)$$

• *ME* in $\langle M, \boldsymbol{g}, \tau_g, \uparrow \rangle$:

$$dF = 0,\quad \underset{g}{\delta} F = -J_e$$

$\Leftrightarrow$

• *Maxwell Equation in* $\langle M, D, \boldsymbol{g}, \tau_g, \uparrow \rangle$: $\partial F = J_e.$ (11)

• *Maxwell Equation in* $\langle M, \nabla, \boldsymbol{g}, \tau_g, \uparrow \rangle$: $\boldsymbol{\partial} F = J_e - \boldsymbol{\mathcal{F}}^{\mathrm{a}} \wedge (\mathfrak{g}_{\mathrm{a}} \underset{g}{\lrcorner} F) - \boldsymbol{\mathcal{F}}^{\mathrm{a}} \underset{g}{\lrcorner} (\mathfrak{g}_{\mathrm{a}} \wedge F).$ (12)

• *GE* in $\langle M, \boldsymbol{g}, \tau_g, \uparrow \rangle$:

$$d\boldsymbol{\mathcal{F}}^{\mathrm{d}} = 0,\quad \underset{g}{\delta}\boldsymbol{\mathcal{F}}_{\mathrm{d}} = -(\mathcal{T}_{\mathrm{d}} + \mathbf{t}_{\mathrm{d}}),$$

$$\boldsymbol{\mathcal{F}}^{\mathrm{d}} = d\mathfrak{g}^{\mathrm{d}}.$$

$\Leftrightarrow$

• *Gravitational Equation in* $\langle M, D, \boldsymbol{g}, \tau_g, \uparrow \rangle$: $\partial\boldsymbol{\mathcal{F}}^{\mathrm{d}} = \mathcal{T}^{\mathrm{d}} + \mathbf{t}^{\mathrm{d}} \Leftrightarrow \partial^2 \mathfrak{g}^{\mathrm{d}} = \mathcal{T}^{\mathrm{d}} + \mathbf{t}^{\mathrm{d}}$ (13)

$\Leftrightarrow (\partial\wedge\partial)\mathfrak{g}^{\mathrm{d}} = -\mathcal{T}^{\mathrm{d}} + \mathbf{t}^{\mathrm{d}} - (\partial\cdot\partial)\mathfrak{g}^{\mathrm{d}} \Leftrightarrow \boldsymbol{\mathcal{R}}^{\mathrm{d}} = -\mathcal{T}^{\mathrm{d}} + \mathbf{t}^{\mathrm{d}} - (\partial\cdot\partial)\mathfrak{g}^{\mathrm{d}} \overset{?}{\Leftrightarrow} \boldsymbol{\mathcal{R}}^{\mathrm{d}} = -\mathcal{T}^{\mathrm{d}} + \mathbf{t}^{\mathrm{d}} + \frac{1}{2} R\mathfrak{g}^{\mathrm{d}}\ (\mathcal{E})$

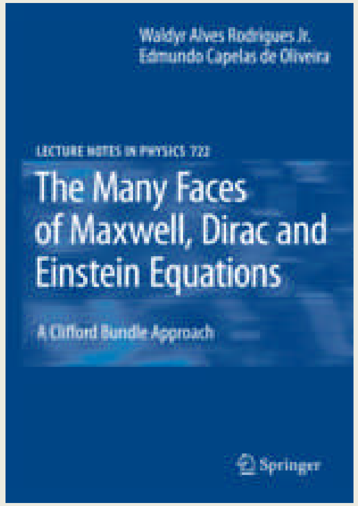
• *Gravitational Equation in* $\langle M, \nabla, \boldsymbol{g}, \tau_g, \uparrow \rangle$: $\boldsymbol{\partial}\boldsymbol{\mathcal{F}}^{\mathrm{d}} = \mathcal{T}^{\mathrm{d}} + \mathbf{t}^{\mathrm{d}} - \boldsymbol{\mathcal{F}}^{\mathrm{a}} \wedge (\mathfrak{g}_{\mathrm{a}} \underset{g}{\lrcorner} \boldsymbol{\mathcal{F}}^{\mathrm{d}}) - \boldsymbol{\mathcal{F}}^{\mathrm{a}} \underset{g}{\lrcorner} (\mathfrak{g}_{\mathrm{a}} \wedge \boldsymbol{\mathcal{F}}^{\mathrm{d}}).$ (14)

The Lagrangian density (Eq.($\mathcal{L}$)) for the gravitational field $\mathfrak{F}^{\mathrm{a}} = d\mathfrak{g}^{\mathrm{a}}$,

$$\mathcal{L}_g = -\frac{1}{2} d\mathfrak{g}^{\mathrm{a}} \wedge \underset{g}{\star} d\mathfrak{g}_{\mathrm{a}} + \frac{1}{2} \underset{g}{\delta}\, \mathfrak{g}^{\mathrm{a}} \wedge \underset{g}{\star} \underset{g}{\delta}\, \mathfrak{g}_{\mathrm{a}} + \frac{1}{4}(d\mathfrak{g}^{\mathrm{a}} \wedge \underset{g}{\star}\, \mathfrak{g}_{\mathrm{a}}) \wedge \underset{g}{\star}(d\mathfrak{g}^{\mathrm{b}} \wedge \underset{g}{\star}\, \mathfrak{g}_{\mathrm{b}}),$$

differs from the Einstein Hilbert Lagrangian density $\mathcal{L}_{EH} = \frac{1}{2} R\tau_g$ by an exact differential, i.e.,

$$\boxed{\mathcal{L}_{EH} - \mathcal{L}_g = -d(\mathfrak{g}^{\mathrm{a}} \wedge \underset{g}{\star} d\mathfrak{g}_{\mathrm{a}}).}$$



This can be seen with some algebra (details in *The Many Faces of ...*) once we recall that

$$\boxed{\mathcal{L}_{EH} = \frac{1}{2} R\tau_g = \frac{1}{2} \mathcal{R}_{\mathrm{cd}} \wedge \underset{g}{\star}\left(\mathfrak{g}^{\mathrm{c}} \wedge \mathfrak{g}^{\mathrm{d}}\right) = \frac{1}{2}(d\omega^{\mathrm{c}}_{\mathrm{d}} + \omega^{\mathrm{c}}_{\mathrm{a}} \wedge \omega^{\mathrm{a}}_{\mathrm{d}}) \wedge \underset{g}{\star}\left(\mathfrak{g}_{\mathrm{c}} \wedge \mathfrak{g}^{\mathrm{d}}\right)}$$

and that the connection 1-form fields of D (the Levi-Civita connection of $\boldsymbol{g}$) can be written as:

$$\boxed{\omega^{\mathrm{cd}} = \frac{1}{2}[\mathfrak{g}^{\mathrm{d}} \underset{g}{\lrcorner} d\mathfrak{g}^{\mathrm{c}} - \mathfrak{g}^{\mathrm{c}} \underset{g}{\lrcorner} d\mathfrak{g}^{\mathrm{d}} + \mathfrak{g}^{\mathrm{c}} \underset{g}{\lrcorner}(\mathfrak{g}^{\mathrm{d}} \underset{g}{\lrcorner} d\mathfrak{g}_{\mathrm{a}})\mathfrak{g}^{\mathrm{a}}].}$$

This warrants the equivalence of the equations:

$$\boxed{d \underset{g}{\star} \mathcal{S}_{\mathrm{d}} + \underset{g}{\star} t_{\mathrm{d}} = -\underset{g}{\star} \mathcal{T}_{\mathrm{d}},} \ (1) \ \Leftrightarrow \quad \boxed{\partial \boldsymbol{\mathcal{F}}^{\mathrm{d}} = \mathcal{T}^{\mathrm{d}} + \mathfrak{t}^{\mathrm{d}}} \ \boxed{\partial^2 \mathfrak{g}^{\mathrm{d}} = \mathcal{T}^{\mathrm{d}} + \mathfrak{t}^{\mathrm{d}}} \ (14)$$

$$\Leftrightarrow \boxed{(\partial \wedge \partial)\mathfrak{g}^{\mathrm{d}} = -\boldsymbol{T}^{\mathrm{d}} + \mathfrak{t}^{\mathrm{d}} - (\partial \cdot \partial)\mathfrak{g}^{\mathrm{d}}} \Leftrightarrow \boxed{\boldsymbol{\mathcal{R}}^{\mathrm{d}} = -\boldsymbol{T}^{\mathrm{d}} + \mathfrak{t}^{\mathrm{d}} - (\partial \cdot \partial)\mathfrak{g}^{\mathrm{d}}} \Leftrightarrow \boxed{\boldsymbol{\mathcal{R}}^{\mathrm{d}} = -\boldsymbol{T}^{\mathrm{d}} + \frac{1}{2} R\mathfrak{g}^{\mathrm{d}} \ (\mathcal{E})}$$

and we have the interesting equation for the energy-momentum 1-forms of the gravitational field

$$\boxed{\mathfrak{t}^{\mathrm{d}} = (\partial \cdot \partial)\mathfrak{g}^{\mathrm{d}} + \frac{1}{2} R\mathfrak{g}^{\mathrm{d}}\ .} \quad (15)$$

The important lesson we learn from this exercise is that Einstein equations can be written in the structure simlpy as:

$$\boxed{d \underset{g}{\star} \mathcal{S}_{\mathrm{d}} + \underset{g}{\star} t_{\mathrm{d}} = -\underset{g}{\star} \mathcal{T}_{\mathrm{d}},} \ (1) \ \Leftrightarrow \boxed{d\boldsymbol{\mathcal{F}}^{\mathrm{d}} = 0,\ \underset{g}{\delta}\, \boldsymbol{\mathcal{F}}_{\mathrm{d}} = -(\mathcal{T}_{\mathrm{d}} + \mathfrak{t}_{\mathrm{d}}), \boldsymbol{\mathcal{F}}^{\mathrm{d}} = d\mathfrak{g}^{\mathrm{d}},}$$

where no connection, no curvature, no torsion, is involved.

Energy-Momentum Conservation

- From the Eq.(7b) ($\underset{g}{\delta} \mathcal{F}_{\mathbf{d}} = -(\mathcal{T}_{\mathbf{d}} + \mathbf{t}_{\mathbf{d}})$) it follows trivially that

$$\boxed{\underset{g}{\delta}(\mathcal{T}_{\mathbf{d}} + \mathbf{t}_{\mathbf{d}}) = 0,} \quad (16)$$

may be interpreted as a *legitimate energy-momentum conservation law* in a teleparallel structure $\langle M, \nabla, \boldsymbol{g}, \tau_g, \uparrow \rangle$ or in the particular teleparallel structure $\langle M, \overset{\circ}{D}, \eta, \tau_\eta, \uparrow \rangle$, the Minkowski spacetime structure.

- In any of those structures we can in an obvious way identify all tangent spaces of M.

Indeed, if $v_x = v^{\mathrm{a}}(x) \boldsymbol{e}_{\mathrm{a}}\big|_x \in \sec T_x M$ and $v_y = v^{\mathrm{a}}(y) \boldsymbol{e}_{\mathrm{a}}\big|_y \in \sec T_y M$ we define the equivalence relation (∥) in TM by $\boxed{v_x = v_y \text{ if and only if } v^{\mathrm{a}}(x) = v^{\mathrm{a}}(y).}$

- We define $\mathbf{v} = [v_x]$. The set of all $\mathbf{v}$ obtained from all the $v_x \in \sec T_x M$ defines a 4-d real vector space $\boldsymbol{V}$.
- We can take as a basis for $\boldsymbol{V}$ the ordered set $\{\boldsymbol{E}_0, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$ with $\boldsymbol{E}_{\mathrm{a}} = [\boldsymbol{e}_{\mathrm{a}}\big|_x]$.
- Thus using Eq.(7) and Stokes theorem we can define the total energy-momentum *vector* of the gravitational plus matter fields by:

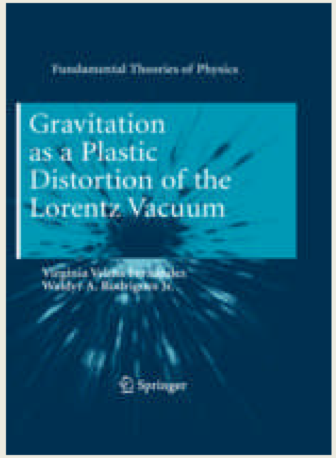
$$\boxed{P := P^{\mathrm{d}} \boldsymbol{E}_{\mathrm{d}}, \qquad P^{\mathrm{d}} := -\frac{1}{8\pi}\int_B \underset{g}{\star}(\mathcal{T}^{\mathrm{d}} + \mathbf{t}^{\mathrm{d}}) = \frac{1}{8\pi}\int_{\partial B} \underset{g}{\star} \mathcal{F}^{\mathrm{d}}.} \quad (17)$$

- Eq.(1) $[d \underset{g}{\star} \mathcal{S}_{\mathrm{d}} + \underset{g}{\star} t_{\mathrm{d}} = -\underset{g}{\star} \mathcal{T}_{\mathrm{d}}]$, permit us to define an alternative conserved energy-momentum law by:

$$\boxed{P' := P'^{\mathrm{d}} \boldsymbol{E}_{\mathrm{d}}, \qquad P'^{\mathrm{d}} := -\frac{1}{8\pi}\int_B \underset{g}{\star}(\mathcal{T}^{\mathrm{d}} + t^{\mathrm{d}}) = \frac{1}{8\pi}\int_{\partial B} \underset{g}{\star} \mathcal{S}^{\mathrm{d}}.} \quad (18)$$

Hamiltonian Formalism

• If we define as usual the canonical momenta associated to the potentials $\mathfrak{g}^{\mathrm{a}}$ by $\mathfrak{p}_{\mathrm{a}} = \partial \mathcal{L}_g / \partial d\mathfrak{g}^{\mathrm{a}} = \underset{g}{\star} \mathcal{S}_{\mathrm{a}}$,



and suppose that this equation can be solved for the $d\mathfrak{g}^{\mathrm{a}}$ as function of the $\mathfrak{p}_{\mathrm{a}}$ we can introduce a *Legendre transformation* with respect to the fields $\mathrm{d}\mathfrak{g}^{\mathrm{a}}$ by

$$\mathbf{L} : (\mathfrak{g}^{\mathrm{a}}, \mathfrak{p}_{\mathrm{a}}) \mapsto \mathbf{L}(\mathfrak{g}^{\mathrm{a}}, \mathfrak{p}_{\mathrm{a}}) = d\mathfrak{g}^{\mathrm{a}} \wedge \mathfrak{p}_{\mathrm{a}} - \mathcal{L}_g(\mathfrak{g}^{\mathrm{a}}, d\mathfrak{g}^{\mathrm{a}}(\mathfrak{p}_{\mathrm{a}})) \quad (19)$$

Put $\mathfrak{L}_g(\mathfrak{g}^{\mathrm{a}}, \mathfrak{p}_{\mathrm{a}}) := \mathcal{L}_g(\mathfrak{g}^{\mathrm{a}}, d\mathfrak{g}^{\mathrm{a}}(\mathfrak{p}_{\mathrm{a}}))$. Observe that defining

$$\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^{\mathrm{a}}, \mathfrak{p}_{\mathrm{a}}) / \boldsymbol{\delta}\mathfrak{g}^{\mathrm{a}} = -d\mathfrak{p}_{\mathrm{a}} - \partial \mathbf{L} / \partial \mathfrak{g}^{\mathrm{a}}, \quad \boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^{\mathrm{a}}, \mathfrak{p}_{\mathrm{a}}) / \boldsymbol{\delta}\mathfrak{p}^{\mathrm{a}} = d\mathfrak{g}_{\mathrm{a}} - \partial \mathbf{L} / \partial \mathfrak{p}^{\mathrm{a}}, \quad (20)$$

we can obtain (details in *Gravitation as a Plastic...*)

$$\boxed{\boldsymbol{\delta}\mathfrak{g}^{\mathrm{a}} \wedge \boldsymbol{\delta}\mathcal{L}_g(\mathfrak{g}^{\mathrm{a}}, d\mathfrak{g}^{\mathrm{a}}) = \boldsymbol{\delta}\mathfrak{g}^{\mathrm{a}} \wedge (\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^{\mathrm{a}}, \mathfrak{p}_{\mathrm{a}}) / \boldsymbol{\delta}\mathfrak{g}^{\mathrm{a}}) + (\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^{\mathrm{a}}, \mathfrak{p}_{\mathrm{a}}) / \boldsymbol{\delta}\mathfrak{p}^{\mathrm{a}}) \wedge \boldsymbol{\delta}\mathfrak{p}^{\mathrm{a}}.} \quad (21)$$

• To define the Hamiltonian form we need something to act the role of time for our manifold, and we choose this "time" to be given by the fow of an arbitrary timelike vector field $\mathbf{Z} \in \sec TM$, such that $g(\mathbf{Z}, \mathbf{Z}) = 1$. Moreover we define $Z = g(\mathbf{Z},) \in \sec \Lambda^1 T^* M \hookrightarrow \mathcal{C}\ell(M, g)$. With this choise the variation $\boldsymbol{\delta}$ is generated by the Lie derivative $\mathscr{L}_{\mathbf{Z}}$.

Cartan's magical formula and some trick algebra (details in *Gravitation as a Plastic...*)

$$\boxed{\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^{\mathrm{a}}, \mathfrak{p}_{\mathrm{a}}) = \mathscr{L}_{\mathbf{Z}}\mathfrak{L}_g = d(Z \underset{g}{\lrcorner} \mathfrak{L}_g) + Z \underset{g}{\lrcorner} (d\mathfrak{L}_g) = d\mathscr{L}_{\mathbf{Z}}\mathfrak{g}^{\mathrm{a}} \wedge \mathfrak{p}_{\mathrm{a}} + \mathscr{L}_{\mathbf{Z}}\mathfrak{g}^{\mathrm{a}} \wedge (\boldsymbol{\delta}\mathfrak{L}_g / \boldsymbol{\delta}\mathfrak{g}^{\mathrm{a}}) + \mathscr{L}_{\mathbf{Z}}\mathfrak{p}_{\mathrm{a}} \wedge (\boldsymbol{\delta}\mathfrak{L}_g / \boldsymbol{\delta}\mathfrak{p}_{\mathrm{a}}),}$$

and using Eq.(21) we have

$$\boxed{d(\mathscr{L}_{\mathbf{Z}}\mathfrak{g}^{\mathrm{a}} \wedge \mathfrak{p}_{\mathrm{a}} - Z \underset{g}{\lrcorner} \mathfrak{L}_g) = \mathscr{L}_{\mathbf{Z}}\mathfrak{g}^{\mathrm{a}} \wedge (\partial \mathcal{L}_g / \partial \mathfrak{g}^{\mathrm{a}}).} \quad (22)$$

Hamiltonian 3-form:

$$\boxed{\mathcal{H}(\mathfrak{g}^{\mathrm{a}}, \mathfrak{p}_{\mathrm{a}}) := \mathscr{L}_{\mathbf{Z}}\mathfrak{g}^{\mathrm{a}} \wedge \mathfrak{p}_{\mathrm{a}} - Z \underset{g}{\lrcorner} \mathfrak{L}_g.} \quad (23)$$

From Eqs.(22) and (23)we have when the field equations are satisfied ($\partial \mathcal{L}_g / \partial \mathfrak{g}^{\mathrm{a}} = 0$) that $\boxed{d\mathcal{H} = 0.}$ (24)

► *$\mathcal{H}$ is a conserved Noether current.*

Quasi - local Energy

- Write: $\boxed{\mathcal{H} = Z^{\mathrm{a}}\mathcal{H}_{\mathbf{a}} + dB}$ (25)
- Some algebra shows that: $\boxed{\mathcal{H}_{\mathbf{a}} = -\boldsymbol{\delta}\boldsymbol{\mathcal{L}}_{g} / \boldsymbol{\delta}\mathfrak{g}^{\mathrm{a}},\ B = Z^{\mathrm{a}}\mathfrak{p}_{\mathbf{a}}.}$ (26)
- Meaning of the boundary term B. Consider an arbitrary spacelike surface σ.

$$\boxed{\mathbf{H} := \frac{1}{8\pi}\int_{\sigma}(Z^{\mathrm{a}}\mathcal{H}_{\mathbf{a}} + dB)} \quad (27)$$

▶When $-\boldsymbol{\delta}\boldsymbol{\mathcal{L}}_{g} / \boldsymbol{\delta}\mathfrak{g}^{\mathrm{a}} = 0$, i.e., the field equations are satisfied we are left with the

the *quasi - local energy*: $\boxed{\mathbf{E} = \frac{1}{8\pi}\int_{\partial\sigma} B\ .}$ (28)

▶ If $\{e_{\mathrm{a}}\}$is a basis for TM such that $\mathfrak{g}^{\mathrm{a}}(e_{\mathrm{b}}) = \delta^{\mathrm{a}}_{\mathrm{b}}$ and if we choose $Z = \boldsymbol{e}_{\mathbf{0}}$

we get recalling that $\mathfrak{p}_{\mathbf{a}} = \underset{g}{\star}\boldsymbol{\mathcal{S}}_{\mathbf{a}}$ that: $\boxed{\mathbf{E} = \frac{1}{8\pi}\int_{\partial\sigma}\underset{g}{\star}\boldsymbol{\mathcal{S}}_{0}\ ,}$ (29)

which we recognize as the same quantity given (when $\mathbf{d} = 0$) by Eq.(18), i.e.,

$$\boxed{P^{\mathrm{d}} := -\frac{1}{8\pi}\int_{B}\underset{g}{\star}(\mathcal{T}^{\mathrm{d}} + t^{\mathrm{d}}) = \frac{1}{8\pi}\int_{\partial B}\underset{g}{\star}\mathcal{S}^{\mathrm{d}}.}$$

Relation with the ADM Energy Concept

- Instead of choosing an arbitrary unit timelike vector field $\mathbf{Z}$ start with a global timelike vector field $\mathbf{n} \in \sec TM$ such that $n = \boldsymbol{g}(\mathbf{n},) = N^2 dt \in \sec \Lambda^1 T^*M \hookrightarrow \mathcal{C}\ell(M,g)$, where N is the *lapse function*.
- Then, $n \wedge dn = 0$ and Frobenius theorem says that n induces a foliation of $M = \mathbb{R} \times \sigma_t$ where σ_t is a spacelike hypersurface σ_t with normal $\mathbf{n}$, and $t = \mathrm{x}^0$, with $\{\mathrm{x}^{\mathbf{a}} = \delta^{\mathbf{a}}_{\mu} \mathrm{x}^{\mu}\}$ coordinates in EPL gauge, i.e., $\eta(d\mathrm{x}^{\mu}, d\mathrm{x}^{\nu}) = \eta^{\mu\nu}$.
- For $A \in \sec \Lambda^1 T^*M \hookrightarrow \mathcal{C}\ell(M,g)$ write $A = \underline{A} + {}^{\perp}A$, with $\underline{A}$ and ${}^{\perp}A$ the tangent and normal components to σ_t. We have: $\boxed{\underline{A} = n \lrcorner_g (dt \wedge A), \quad {}^{\perp}A = dt \wedge A_{\perp}, \; A_{\perp} = n \lrcorner A.}$ (30)
- Put $\boxed{\underline{d}A := dt \lrcorner_g (n \wedge dA).}$ Cartan's magical formula gives: $\boxed{dA = dt \wedge (\mathscr{L}_{\mathrm{n}} \underline{A} - \underline{d}A_{\perp}) + \underline{dA}}$ (31)
- First fundamental form on σ_t : $\boxed{\boldsymbol{m} = -\boldsymbol{g} + \boldsymbol{n} \otimes \boldsymbol{n} = \underline{\mathfrak{g}}^{\mathrm{i}} \otimes \underline{\mathfrak{g}}_{\mathrm{i}}, \; \boldsymbol{n} = \frac{n}{N}.}$ (32) $\boxed{\underset{m}{\star} \underline{A} := \underset{g}{\star} (\boldsymbol{n} \wedge \underline{A}).}$ (33)
- Writting $\boxed{\mathcal{L}_g(\mathfrak{g}^{\mathrm{a}}, d\mathfrak{g}^{\mathrm{a}}) = dt \wedge \mathcal{K}_g(n^{\mathrm{i}}, \underline{d}n^{\mathrm{i}}, \underline{\mathfrak{g}}^{\mathrm{i}}, \underline{d\,\mathfrak{g}}^{\mathrm{i}}, \mathscr{L}_{\mathrm{n}} \underline{\mathfrak{g}}^{\mathrm{i}})} \Rightarrow \boxed{\mathcal{H}(\underline{\mathfrak{g}}^{\mathrm{i}}, \underline{\mathfrak{p}}_{\mathrm{i}}) = \mathscr{L}_{\mathrm{n}} \underline{\mathfrak{g}}^{\mathrm{i}} \wedge \underset{m}{\star} \underline{\mathfrak{p}}_{\mathrm{i}} - \mathcal{K}_g,}$ (34)
  - Some (trick) algebra gives $\boxed{\mathcal{H} = n^{\mathrm{i}} \mathcal{H}_{\mathrm{i}} + \underline{d}B', \; \mathcal{H}_{\mathrm{i}} = -\underline{(\boldsymbol{\delta} \mathcal{L}_g / \boldsymbol{\delta} \mathfrak{g}^{\mathrm{i}})} = -\boldsymbol{\delta} \mathcal{K}_g / \boldsymbol{\delta} n^{\mathrm{i}}, \; B' = -N \underline{\mathfrak{g}}_{\mathrm{i}} \wedge \underset{m}{\star} \underline{d\,\mathfrak{g}}^{\mathrm{i}},}$ (35)
- When $\boldsymbol{\delta} \mathcal{L}_g / \boldsymbol{\delta} \mathfrak{g}^{\mathrm{i}} = 0$, we get exactly the *ADM* energy $\boxed{\mathbf{E}_{ADM} := -\frac{1}{8\pi} \int_{\partial \sigma_t} N \underline{\mathfrak{g}}_{\mathrm{i}} \wedge \underset{m}{\star} \underline{d\,\mathfrak{g}}^{\mathrm{i}},}$ (36)
- Indeed, take $\partial \sigma_t$ a 2-sphere at infinity. Then, $\underline{\mathfrak{g}}_{\mathrm{i}} = h_{\mathrm{ij}} \underline{d}\mathrm{x}^{\mathrm{j}}$, $h_{\mathrm{ij}} - \delta_{\mathrm{ij}} \to 0, N \to 1,$

$\underline{\mathfrak{g}}_{\mathrm{i}} \wedge \underset{m}{\star} \underline{d\,\mathfrak{g}}^{\mathrm{i}}, = h^{\mathrm{ij}} (\partial h_{\mathrm{ij}} / \partial \mathrm{x}^{\mathrm{k}} - \partial h_{\mathrm{ik}} / \partial \mathrm{x}^{\mathrm{j}}) \underset{m}{\star} \underline{\mathfrak{g}}^{\mathrm{k}}$ and $\boxed{\mathbf{E}_{ADM} = -\frac{1}{8\pi} \int_{\partial \sigma_t} (\partial h_{\mathrm{ij}} / \partial \mathrm{x}^{\mathrm{k}} - \partial h_{\mathrm{ik}} / \partial \mathrm{x}^{\mathrm{j}}) \underset{m}{\star} \underline{\mathfrak{g}}^{\mathrm{k}}}$ (37)

$\mathbf{E}_{ADM} = \mathbf{E}'$ for Isolated Systems

If we choose $n = \mathfrak{g}^0$ it may happen that $\mathfrak{g}^0 \wedge d\mathfrak{g}^0 \neq 0$, and thus it does not determine a spacelike hypersuperface σ_t. However, all algebraic calculations up to Eq.(35) above are valid (and of course $\mathfrak{g}^{\mathrm{i}} = \underline{\mathfrak{g}}^{\mathrm{i}}$). So, if we take a spacelike hypersurface σ such that at spatial infinity the e_{i} $(\mathfrak{g}^{\mathrm{i}}(e_{\mathrm{j}}) = \delta^{\mathrm{i}}_{\mathrm{j}})$ are tangent to σ, and $e_0 \to \partial/\partial t$ is orthogonal to σ, then we have $\mathbf{E}' = \mathbf{E}$ since in this case recalling Eq.(2) for $\mathbf{d} = 0$, i.e.,

$$\boxed{\underset{g}{\star}\mathcal{S}_0 = -\mathfrak{g}_{\mathrm{a}} \wedge \underset{g}{\star}(\mathfrak{g}_0 \wedge d\mathfrak{g}^{\mathrm{a}}) + \frac{1}{2}\mathfrak{g}_0 \wedge \underset{g}{\star}(d\mathfrak{g}^{\mathrm{a}} \wedge \mathfrak{g}_{\mathrm{a}})}$$

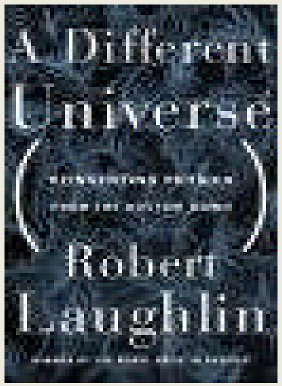
we see that

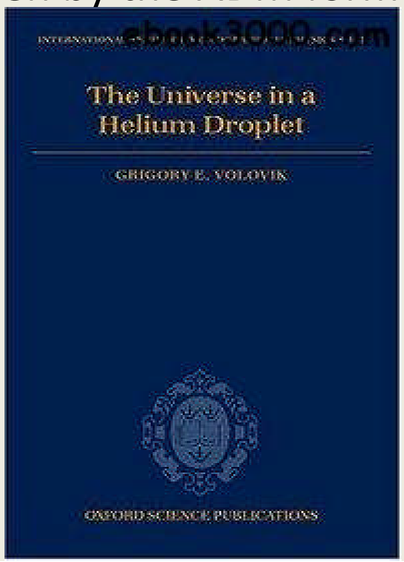
$$\boxed{-N\underline{\mathfrak{g}}_{\mathrm{i}} \wedge \underset{m}{\star}\underline{d}\underline{\mathfrak{g}}^{\mathrm{i}} \to -\mathfrak{g}_{\mathrm{i}} \wedge \underset{g}{\star}(\mathfrak{g}^0 \wedge d\mathfrak{g}^{\mathrm{i}})} \quad (38)$$

is the asymptotic value of $\underset{g}{\star}\mathcal{S}^0$ (taking into account that at spatial infinity $d\mathfrak{g}^0 = 0$).

Conclusions

- We recalled that a gravitational field generated by a given energy-momentum tensor can be represented by distinct geometrical structures and if we prefer, we can even dispense all those geometrical structures and simply represent the gravitational field as a field in the Faraday's sense living in Minkowski spacetime. The explicit Lagrangian density for this theory has been given and the equations of motion presented in a Maxwell like form and shown to be equivalent to Einstein's equations in a precise mathematical sense. We hope that our study clarifies the real difference between mathematical models and physical reality and leads people to think about the real physical nature of the gravitational field (and also of the electromagnetic field as suggested, e.g., by the works of Laughlin and Volikov. We discussed also an Hamiltonian formalism for our theory and the concepts of energy defined by Eq.(29) and the one given by the ADM formalism.





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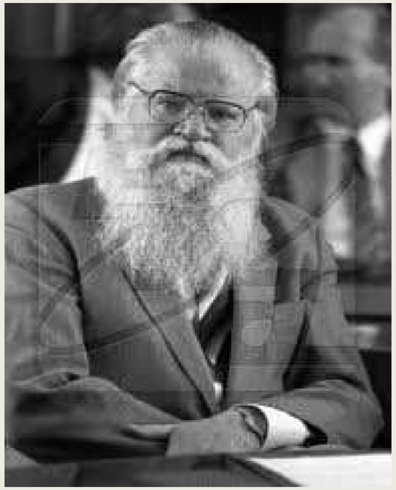

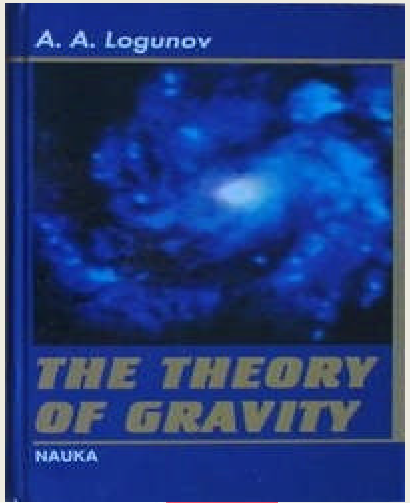
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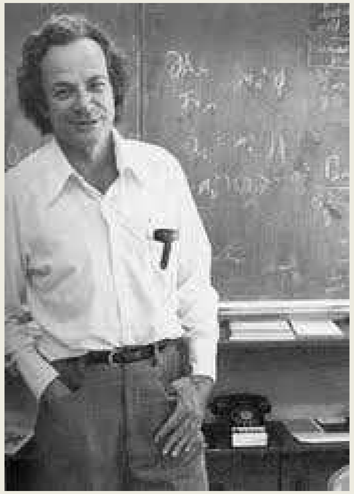

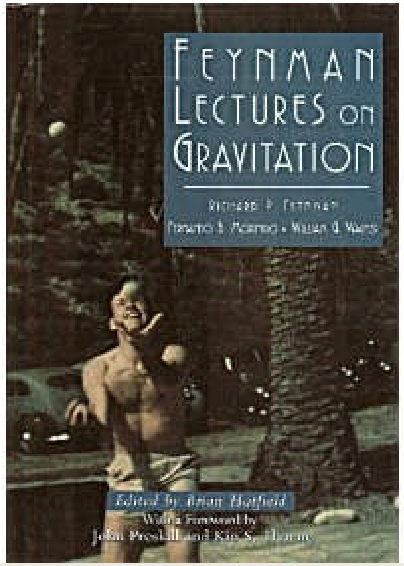
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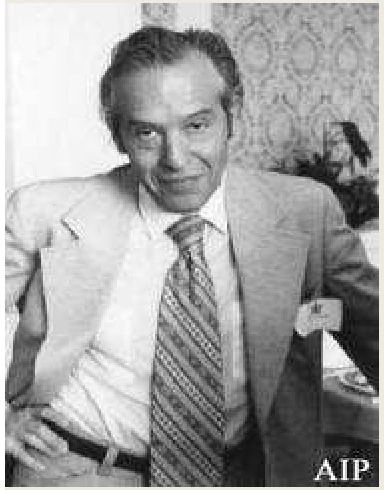
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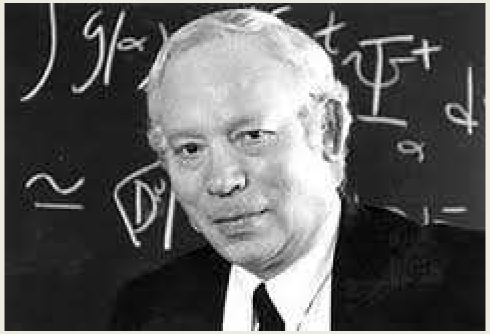

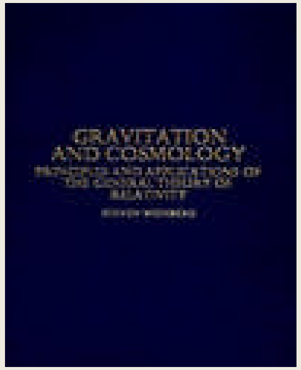
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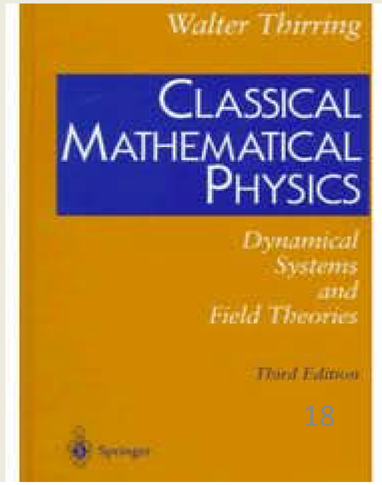
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