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Pair and impair, even and odd form fields, and electromagnetism

Roldão da Rocha[1,*] and **Waldyr A. Rodrigues Jr.**[2,**]

[1] Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, 09210-170, Santo André, SP, Brazil

[2] Institute of Mathematics, Statistics and Scientific Computation, IMECC-UNICAMP CP 6065, 13083-859 Campinas, SP, Brazil

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In this paper after reviewing the Schouten and de Rham definition of *impair* and *pair* differential form fields (not to be confused with differential form fields of even and odd grades) we prove that in a *relativistic spacetime* it is possible (despite claims in contrary) to coherently formulate electromagnetism (and we believe any other physical theory) using only pair form fields or, if one wishes, using pair and impair form fields together, in an appropriate way. Those two distinct descriptions involve only a mathematical choice and do not seem to lead to any observable physical consequence if due care is taken. Moreover, we show in details that a formulation of electromagnetic theory in the Clifford bundle formalism of differential forms where the two Maxwell equations of the so called metric-free approach becomes a single equation is compatible with both formulations of electromagnetism just mentioned above. In addition we derive directly from Maxwell equation the density of force (coupling of the electromagnetic field with the charge current) that is a postulate in the free metric approach to electromagnetism. We recall also a formulation of the engineering version of Maxwell equations using electric and magnetic fields as objects of the same nature, i.e., without using polar and axial vectors.

1 Introduction

Cartan has popularized the use of differential forms which he apparently introduced in 1899 [1], and which are now indispensable tools in several mathematical and physical theories[1]. What is less known among physicists is that those objects come out in two versions, *pair* and *impair* differential forms (also called by some authors pseudo-forms or twisted forms), a concept which has its origin in the Heaviside formulation of electromagnetic theory in terms of *polar* and *axial* vector fields. Rigorously speaking, pair and impair forms are sections of different bundles[2], but here to motivate our presentation we may say that pair forms living on an oriented spacetime are invariant under change of the coframe basis orientation (related to a fixed spacetime orientation) in which they are expressed[3] – in particular, pair 0-forms are *scalar* functions

* E-mail: roldao.rocha@ufabc.edu.br
** Corresponding author E-mail: walrod@ime.unicamp.br

[1] In particular, it seems that Cartan applied differential forms in the formulation of electromagnetism for the first time in [2].

[2] Pair forms are sections of the exterior algebra bundle ΛT^*M and impair forms are sections the bundle $\Lambda_- T^*M \simeq L(M) \otimes \Lambda T^*M$ where $L(M)$ is a line bundle called the orientation bundle of M. Some details are given below.

[3] The orientation of a given coframe basis is not to be confused with the orientation of the manifold (part of the structure defining a spacetime) which is given by an *arbitrary* choice of a volume form. See below for details.

– whereas impair forms change sign under change of the coframe basis orientation in which they are expressed, and in particular *impair* 0-forms are also known as *pseudoscalar* functions.

A definition[4] of such pair and impair differential forms has been originally introduced by de Rham [3] (but see also [4, 5])[5] and will be recalled below.

Of course, the theory of differential forms has been applied by many authors in the formulation of different physical theories (see, e.g., [7]), and in particular in electromagnetism. However, the formulations of that theory appearing, e.g., [8–18] make use only of pair differential forms[6]. It must be said that for those authors, the arena where charged particles and the electromagnetic field interact is a *Lorentzian spacetime*, that as well known is an *oriented* manifold[7]. On the other side authors like, e.g., [19–33] explicitly claim that impair forms are absolutely necessary for a consistent formulation of electromagnetism even in an *oriented* spacetime manifold and mainly if the spacetime is a bare manifold devoid of metric and affine structure[8]. Eventually, the main *argument* of the majority of those authors is that the current 3-form must be impair for otherwise its integral over an oriented 3-chain (which gives the value of the charge in that region) does depend on the orientation chosen, a conclusion that those authors consider an absurd.

Moreover it must be said that the presentation of the differential equations of electromagnetism using the Clifford bundle formalism [7] uses only pair differential forms and, if the charge argument is indeed correct, it seems to imply that the Clifford bundle cannot be used to describe electromagnetism or any other physical theory. So, we must discuss in a thoughtful way the claims of [19–24, 27–33, 38] and indeed, the main purpose of the present paper is to do that by showing that in a *relativistic spacetime*[9] the electromagnetic theory[10] can be rigorously presented with all fields involved being pair form fields. Of course, a presentation of electromagnetism in a oriented (even if bare) spacetime using appropriate pair and impair form fields is also *correct*, but as it will become clear below it seems to be nothing more than a simply *option*, not a *necessity*. Moreover, we show that contrary to a first expectation, the formulation of electromagnetism in the Clifford bundle [7] of (*pair*) form fields is automatically compatible with each one of those mentioned formulations of the theory, i.e., starting from Maxwell equations formulated as a single equation in the Clifford bundle, we can show that from that equation we can either obtain as a result of a straightforward mathematical *choice* two equations involving only pair forms *or* two equations such that one uses pair forms and the other impair forms.

This paper is organized as follows: in Sect. 2 we introduce the nature of the spacetime manifold used in the formulation of relativistic physical theories and recall Maxwell equations formulated with *pair* differential forms on Minkowski spacetime, calling the reader's attention to the fact that Maxwell equations describe only one aspect of electromagnetism, which is a theory describing the interaction of the electromagnetic field with charged particles (see specially Sect. 6). Moreover, we emphasize that although only the manifold structure of M is enough for the writing of Maxwell equations, the remaining objects which defines the Minkowski spacetime structure play a fundamental role in the theory, as showed on several

4 We will give an alternative equivalent definition below.

5 It must be said that in de Rham's discussion of cohomology, impair forms have disappeared. In [6], this question is suitably studied by the investigation of the Grassmann and Clifford algebras over Peano spaces, introducing their respective associated extended algebras, and exploring these concepts also from the counterspace viewpoint. It was shown that the de Rham cochain, generated by the codifferential operator related to the regressive product, is composed by a sequence of exterior algebra homogeneous subspaces that are subsequently pair and impair.

6 Some authors as e.g., [19] avoid the use of pair and impair forms by using instead (pair) multivector fields and pair forms.

7 Almost all the authors in [8–18] even do not mention impair differential forms in their books and the few that mention those objects only say that they are necessary for a consistent integration theory on non oriented manifolds.

8 The idea of developing electromagnetism using a manifold devoid of metric and affine structure is very old and appears in [34] and [35]. A complete set of references on the subject up to 1960 is given in [36]. Such an approach to electromagnetism has also been used in [37] and now is advocated by many authors, see specially [23] (and of course, the arXiv) for modern references.

9 The concept of a relativistic spacetime as used in this paper is recalled in Sect. 2.

10 This includes even the case of regions involving a non dispersive medium that can be described by effective Lorentzian spacetimes [30].

times in different sections of the paper. Section 3 is dedicated to the definition of the pair volume 4-form and the pair Hodge star operator. Section 4 defines *impair* differential forms and in particular emphasizes the difference between the pair and the impair volume forms and the pair and impair Hodge star operators. Section 4 also discusses the fundamentals of electromagnetism in a medium and proves that, contrary to some claims [39], the recent discovery that the constitutive extensor of Cr_2O_3 has a term proportional to the Levi-Civita symbol in no way implies that this discovery is the proof that impair forms must be used in the formulation of electromagnetism. Section 5 recalls the Clifford bundle formulation of Maxwell equation[11], proving as already mentioned that it is compatible with those two formulations (only pair and pair and impair) of that equations. Section 6 shows how the force density that is postulated in the presentation of electromagnetism in [23] is directly contained in Maxwell equation. Moreover we show in Sect. 6 that the equation (which contains the force density) describing the interaction of the charged particles with the field automatically knocks down the *charge argument* mentioned above. In Sect. 7 using the Pauli algebra bundle we present the engineering formulation of electromagnetism in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ and emphasize that in this formulation which necessarily needs a *choice* of a volume element we do *not* need to introduce the so called *axial* vector fields and moreover that the circulation of the magnetic field around a (very long) wire conducting current is *conventional*. Finally in Sect. 8 we present our concluding remarks.

2 Nature of the spacetime manifold and of the electromagnetic field

Every physical theory starts by modeling the arena (spacetime) where physical phenomena are supposed to happen. It is a well known fact that when gravitation can be neglected, the motion (classical or quantum) of particles and fields occurs in an arena which is modeled by Minkowski spacetime, i.e., a structure $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$, where M is a 4-dimensional manifold diffeomorphic to $\mathbb{R}^4$, $\boldsymbol{g} \in \sec T_0^2 M$ is a Lorentzian metric, D is the Levi-Civita connection of $\boldsymbol{g}$ (i.e., $\mathbf{T}(D) = 0$, where $\mathbf{T}$ is the torsion tensor associated to the connection D), $\mathbf{R}(D)$ is the curvature tensor associated with D, $\tau_{\boldsymbol{g}} \in \sec \Lambda^4 T^* M$ is the metric volume element, i.e., a *pair*[12] 4-form defining a spacetime orientation and $\uparrow$ denotes time orientation[13].

Classical electromagnetism according to Feynman is the theory which describes the interaction of objects called charged particles and the electromagnetic field $F \in \sec \Lambda^2 T^* M$ called field strength. For the purposes of this paper a charged particle is described by a triple (m, q, σ), where (conventionally) $m \in \mathbb{R}^+$ is the *mass* parameter and based on experimental facts q (the charge) is a non null integral multiple of an elementary charge denoted $|e|$. It is extremely important to keep in mind for the objectives of the present paper that the sign of q to be attributed to any charge depends on a *convention*, which will be scrutinized latter (Remark 17 and Sect. 4.3). Moreover, $\sigma : \mathbb{R} \to \mathbb{M}$ is timelike curve pointing to the future[14]. We parametrize σ in such a way that $\boldsymbol{g}(\sigma_*, \sigma_*) = 1$ and define a 1-form field over σ, denoted by $v = \boldsymbol{g}(\sigma_*)$. Given a finite collection of particles $(m^{(i)}, q^{(i)}, \sigma^{(i)})$, $i = 1, 2, \ldots, n$, we define the *current* for the i-particle as the 1-form field $J^{(i)} = q^{(i)}\, v^{(i)}$ over σ. The total current of the system is given by

$$J = \sum\nolimits_i J^{(i)} \tag{1}$$

[11] No misprint here. To know why look at Eq. (59).

[12] See below for the definition of *pair* and *impair* forms. We emphasize here that the association of an orientation to a pair form is the one used , e.g., in [40].

[13] More details may be found, e.g. in [7, 42].

[14] This being the reason why we suppose that spacetime is time orientable.

which support are the set of timelike lines $\cup_i \sigma^{(i)}$. If we introduce a global coordinate chart for M with coordinates $\{\mathrm{x}^\mu\}$ in the Einstein-Lorentz-Poincaré gauge[15] then we can write

$$J = \sum\nolimits_i J_\mu^{(i)} \gamma^\mu, \tag{2}$$

$$J_\mu^{(i)} = \eta_{\mu\nu} q^{(i)} \int \delta^{(4)}(\mathrm{x}^\beta - \mathrm{x}^\beta \circ \sigma^{(i)}(s_{(i)})) \frac{d\mathrm{x}^\nu \circ \sigma^{(i)}(s_{(i)})}{ds_{(i)}} ds_{(i)}, \tag{3}$$

with $s_{(i)}$ being the proper time along $\sigma^{(i)}$. Before going on we must say that if the density of particles is very large we may eventually approximate J by a continuous section of $\Lambda^1 T^* M$ or at least by a de Rham current [3]. It is an empirical fact that F is closed, i.e., $dF = 0$ and moreover[16], $\mathbf{J} = \underset{\tau_g}{\star} J \in \sec \Lambda^3 T^* M$ is exact, i.e., $\mathbf{J} = -dG$ for $G \in \sec \Lambda^3 T^* M$, called the excitation field. Those empirical observations are written as

$$dF = 0,\ dG = -\mathbf{J}, \tag{4}$$

and known as Maxwell equations.

Remark 1. Before we proceed it must be said that if we forget the fact that the carriers of charges are particles and simply suppose that experimentally all we have is a $\mathbf{J} \in \sec \Lambda^3 T^* M$ that is conserved (i.e., $d\mathbf{J} = \mathbf{0}$), then supposing that the manifold where $\mathbf{J}$ lives is star-shaped, we have $dG = -\mathbf{J}$. Using moreover the fact that $dF = 0$ (meaning that magnetic monopoles do not exist) it is a mathematical fact that the system of differential equations given by Eq. (4) does need for its writing only the structure of the base manifold structure M, i.e., it does not need the additional objects $(\boldsymbol{g}, D, \tau_g, \uparrow)$ entering the structure of Minkowski spacetime[17]. However, electromagnetism is *not* only Maxwell equations, we must yet specify the way that two currents $\mathbf{J}^{(1)}, \mathbf{J}^{(1)} \in \sec \Lambda^1 T^* M$ interact. And to do this we shall need to use the additional structure, as we shall see.

To proceed with our presentation of electromagnetism we must recall that as it is well known the metric tensor can be used to give a Clifford bundle structure to $\Lambda T^* M = \oplus_{p=0}^4 \Lambda T^* M$, which will be called (for reasons to be explained below) the *pair* bundle of differential forms. The Clifford bundle of nonhomogeneous differential forms is denoted by[18] $\mathcal{C}\ell(M, g)$, where $g \in \sec T_2^0 M$ denotes the metric of the cotangent bundle, such that for any arbitrary basis $\{e_\mu\}$ of $TU \subseteq TM$ and dual basis $\{\boldsymbol{\theta}^\mu\}$ of $T^*U \subseteq T^*M$ (U are open sets in M), $\theta^\mu \in \sec \Lambda^1 T^* U \subseteq \sec \Lambda T^* M \hookrightarrow \sec \mathcal{C}\ell(M, g)$, we have $\boldsymbol{g} = g_{\mu\nu} \theta^\mu \otimes \theta^\nu$, $g = g^{\mu\nu} e_\mu \otimes e_\nu$ and $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda$.

Remark 2. We recall that any section of $\Lambda^r T^* M$ is said to be a r-graded form field (or r-form for short). Sometimes it is said to be of *even* or *odd* grade, depending on whether r is even or odd. This classification is not to be confused to the concept of de Rham *pair* and *impair* forms, to be introduced below.

[15] Given a Minkowski spacetime structure, global coordinates $\{\mathrm{x}^\mu\}$ for $M \simeq \mathbb{R}^4$ are said to be in the Einstein-Lorentz-Poincaré gauge (*ELPG*) if and only if the following conditions hold: $\boldsymbol{g} = \eta_{\mu\nu} d\mathrm{x}^\mu \otimes d\mathrm{x}^\nu$, $D_{\frac{\partial}{\partial \mathrm{x}^\mu}} \frac{\partial}{\partial \mathrm{x}^\nu} = 0$. Of course, as well known, there exists an infinity of coordinate functions related by Poincaré transformations satisfying these conditions. We shall write in what follows $\gamma^\mu := d\mathrm{x}^\mu$ and $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$.

[16] The symbol $\underset{\tau_g}{\star}$ means the *pair* Hodge dual operator, and its definition is given below.

[17] This has been originally observed by Cartan in [2].

[18] Details on the construction of $\mathcal{C}\ell(M, g)$ may be found, e.g., in [7].

2.1 Energy-momentum 1-form and energy-momentum tensor for the system of charged particles

For use in Sect. 6 we define now the energy-momentum 1-form for a charged particle $(m^{(i)}, q^{(i)}, \sigma^{(i)})$ as the 1-form field $p^{(i)}$ over $\sigma^{(i)}$ given by

$$p^{(i)} = m^{(i)} v^{(i)}, \tag{5}$$

and it is obvious that $p^{(i)} \cdot p^{(i)} = (m^{(i)})^2$. In an inertial frame[19] $\mathbf{I} = \frac{\partial}{\partial \mathrm{x}^0}$ associated to the coordinates $\{\mathrm{x}^\mu\}$ for M in *ELPG* at time $\mathrm{x}^0 = t$, the particles will occupy different spacetime points $(t, \mathrm{x}^1_{(i)}(t), \mathrm{x}^2_{(i)}(t), \mathrm{x}^3_{(i)}(t))$. We can define the total momentum of the particles at time t only if it is *licit* to sum distinct 1-forms at different tangent spaces of M. This, of course requires an *absolute parallelism* and here is then a place where the flat connection D that was introduced in the structure of Minkowski spacetime becomes necessary. It permits us to write the total momentum of the particles at time t as

$$P(t) = \sum\nolimits_i p^{(i)}(t), \tag{6}$$

a necessary concept needed in order to be possible to talk about energy-momentum conservation for the system of particles and the electromagnetic field (see Sect. 6). Besides the momentum 1-form of the particles we shall need also to introduce the energy-momentum 1-forms $\mathbf{T}^\alpha \in \sec \Lambda^1 T^* M$ for the system of charged particles. We have:

$$\begin{aligned} \mathbf{T}^\alpha &= \mathbf{T}^{\alpha\beta} \gamma_\beta, \\ \mathbf{T}^{\alpha\beta} &= \sum\nolimits_i \eta^{\alpha\mu} \int p^{(i)}_\mu(s) \frac{d}{ds} \mathrm{x}^\beta \circ \sigma^{(i)}(s_{(i)}) \delta^4(\mathrm{x}^\kappa - \mathrm{x}^\kappa \circ \sigma^{(i)}(s_{(i)})) ds_{(i)}. \end{aligned} \tag{7}$$

3 The pair metric volume element τ_g

First introduce an arbitrary $\boldsymbol{g}$-orthonormal basis $\{\mathbf{e}_\alpha\}$ for TM and corresponding dual basis $\{\boldsymbol{\theta}^\alpha\}$ for T^*M. Then, $\boldsymbol{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = \eta_{\alpha\beta}$ and $g(\boldsymbol{\theta}^\alpha, \boldsymbol{\theta}^\beta) = \boldsymbol{\theta}^\alpha \cdot \boldsymbol{\theta}^\beta = \eta^{\alpha\beta}$ and $\boldsymbol{\theta}^\alpha(\mathbf{e}_\beta) = \delta^\alpha_\beta$, where the matrix with entries $\eta_{\alpha\beta}$ and the matrix with entries $\eta^{\mu\nu}$ are equal to the diagonal matrix $\mathrm{diag}(1, -1, -1, -1)$. We define a *pair* metric volume[20] $\tau_g \in \sec \Lambda^4 T^* M$ by

$$\tau_g := \boldsymbol{\theta}^0 \wedge \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2 \wedge \boldsymbol{\theta}^3 \tag{8}$$

Remark 1 An *orientation* for M as we already said above is a free choice of an arbitrary volume element.[21] Before proceeding let us introduce arbitrary coordinates $\{x^\mu\}$ for $U \subset M$ and $\{x'^\mu\}$ for $U' \subset M$, $U \cap U' \neq \varnothing$ such that

$$\boldsymbol{\theta}^\alpha = h^\alpha_\mu dx^\mu, \boldsymbol{\theta}^\alpha = h'^\alpha_\mu dx'^\mu. \tag{9}$$

Let $\mathbf{h}$ and $\mathbf{h}'$ the matrices with entries h^α_μ and h'^α_μ. Then, e.g.,

$$\det(g_{\alpha\beta}) = (\det \mathbf{h})^2 \det(\eta_{\alpha\beta}), \tag{10}$$

[19] An inertial frame is defined as a time like vector field $\mathbf{I}$ such that $\boldsymbol{g}(\mathbf{I}, \mathbf{I}) = 1$ and $D\mathbf{I} = \mathbf{0}$. More details if need may be found, e.g., at [7].

[20] The impair volume elements is defined in Sect. 4.1.

[21] Of course, an arbitrary manifold M, even if orientable, when equipped with an arbitrary Lorentzian metric field $\mathtt{g}$ does not in general admit a global $\mathtt{g}$-orthonormal cotetrad field, so, in this case the introduction of $\tau_{\mathtt{g}}$ is a little more complicated [9, 11]. However, all manifolds M part of a Lorentzian spacetime structure that admits spinor fields have a global $\mathtt{g}$-orthonormal cotetrad field. This is a result from a famous theorem due to Geroch [43].

and

$$\sqrt{\left|\det(g_{\alpha\beta})\right|} = |\det \mathbf{h}| \sqrt{\left|\det(\eta_{\alpha\beta})\right|} = |\det \mathbf{h}| \tag{11}$$

The expression of τ_g in the bases $\{dx^\mu\}$ and $\{dx'^\mu\}$ are respectively

$$\begin{aligned}\tau_g &= \frac{1}{4!}\tau_{i_0\ldots i_3} dx^{i_0} \wedge \cdots \wedge dx^{i_3} = \tau_{0123} dx^0 \wedge \cdots \wedge dx^3 \\ &= \frac{\det \mathbf{h}}{|\det \mathbf{h}|}\sqrt{\left|\det(g_{\alpha\beta})\right|} dx^0 \wedge \cdots \wedge dx^3, \end{aligned} \tag{12}$$

and

$$\tau_g = \frac{1}{4!}\tau'_{j_0\ldots j_3} dx'^{j_0} \wedge \cdots \wedge dx'^{j_3} = \tau'_{0123} dx'^0 \wedge \cdots \wedge dx'^3. \tag{13}$$

Now writing $L^{i_p}_{j_p} = \frac{\partial x^{i_p}}{\partial x'^{j_p}}$, $\det L = \det(\frac{\partial x^i}{\partial x'^j})$ we have (remember that $\tau_{i_0\ldots i_3} = \varepsilon^{0123}_{i_0\ldots i_3}\tau_{0123} = \varepsilon^{0123}_{i_0\ldots i_3}\frac{\det \mathbf{h}}{|\det \mathbf{h}|}\sqrt{\left|\det(g_{ij})\right|}$)

$$\tau'_{j_0\ldots j_3} = L^{i_0}_{j_0} \ldots L^{i_3}_{j_3}\tau_{i_0\ldots i_3}. \tag{14}$$

Also, since $\sqrt{\left|\det(g'_{ij})\right|} = |\det \Lambda| \sqrt{\left|\det(g_{ij})\right|}$, we end with

$$\tau'_{0123} = \det L \tau_{0123} := \Delta^{-1}\tau_{0123} \tag{15}$$

$$= \frac{\det \mathbf{h}}{|\det \mathbf{h}|}\frac{\det L}{|\det L|}\sqrt{\left|\det(g'_{ij})\right|}. \tag{16}$$

Remark 3. We want to emphasize here that with the choice $\frac{\det \mathbf{h}}{|\det \mathbf{h}|} = +1$, the coordinate expression for τ_g in the basis $\{dx^\mu\}$ becomes the one appearing in almost all textbooks, i.e., $\sqrt{\left|\det(g_{ij})\right|} dx^0 \wedge \cdots \wedge dx^3$. However the coordinate expression for τ_g in the basis $\{dx'^\mu\}$ is $\sqrt{\left|\det(g'_{ij})\right|} dx'^0 \wedge \cdots \wedge dx'^3$ only if $\frac{\det L}{|\det L|} = +1$. The omission of the factor $\frac{\det L}{|\det L|}$ in the textbook presentation of τ_g is the source of a big confusion and eventually responsible for a statement saying that the volume element must be an impair 4-form. An *impair* volume element is an object different from τ_g and will be introduced in Sect. 4.1.

Comparing Eq. (15) to Eq. (8.1) of Schouten's book [4] we see the reason why a quantity that "transforms" like in Eq. (15) is called a *scalar-Δ-density of weight* 1. Despite being fan of Schouten's book, the authors think that such a nomenclature may induce confusion, unless expressed in a coordinate free way as, e.g., done in [40, 41, 46].

3.1 The pair Hodge star operator

A pair metric volume element τ_g permits us to define an isomorphism between $\Lambda^p T^*M \hookrightarrow \mathcal{C}\ell(M,g)$ and $\Lambda^{4-p} T^*M \hookrightarrow \mathcal{C}\ell(M,g)$, given by

$$\begin{aligned} \underset{\tau_g}{\star} : \ & \Lambda^p T^*M \to \Lambda^{4-p} T^*M \\ & A_p \mapsto \underset{\tau_g}{\star} A_p := \tilde{A}_p \tau_g \end{aligned} \tag{17}$$

In Eq. (17) $\tilde{A}_p \tau_g$ means the Clifford product between the Clifford fields $\tilde{A}_p$ and τ_g, and $\tilde{A}_p$ is the reverse[22] of A_p. Let $\{x^\mu\}$ be global coordinates in the *ELPG* and $\{\gamma^\mu = dx^\mu\}$ an orthonormal cobasis, i.e., $g(\gamma^\mu, \gamma^\nu) := \gamma^\mu \cdot \gamma^\nu = \eta^{\mu\nu}$.

In this case we *can* write $\tau_g = \gamma^5 = \gamma^0 \wedge \gamma^1 \wedge \gamma^2 \wedge \gamma^3 = \gamma^0\gamma^1\gamma^2\gamma^3$ and the calculation of the action of the Hodge dual operator on a p-form becomes an elementary algebraic operation[23]. We also suppose that $\tau_g = \gamma^5$ defines a *positive orientation* (also called *right handed orientation*), and it is trivial to verify that[24]

$$\tau_g \tau_g = \tau_g \cdot \tau_g = (\gamma^5)^2 = -1 \tag{18}$$

Before we proceed, recall that we can show trivially that the definition given by Eq. (17) is equivalent to the standard one, i.e., for any $A_p, B_p \in \sec \Lambda^p T^*M$, it follows that

$$B_p \wedge \underset{\tau_g}{\star} A_p = (B_p \cdot A_p)\tau_g \tag{19}$$

Remark 4. As defined, the object $A'_p = \underset{\tau_g}{\star} A_p$ is a legitimate *pair* form, although it depends, as it is obvious from Eq. (17), of the chosen orientation τ_g. Some authors (like e.g., [23]) assert that the Hodge star operator maps a pair form into an impair one (see the definition of impair forms below). What does this statement mean given our definition that the (pair) Hodge star operator changes an even grade form to an odd grade form (and vice versa)? It means that an impair Hodge operator that maps a pair form into an impair one can also be defined and of course is a concept different from the one just introduced. The impair Hodge operator will be presented and discussed in Sect. 4.1. To avoid any possible confusion on this issue, let us bethink that there may two different Hodge operators associated with the same metric $\boldsymbol{g}$. Indeed, given the metric $\boldsymbol{g}$ and a (pair) metric volume 4-form $\tau'_g \neq \tau_g$ with $\tau'^2_g = -1$ we may define another Hodge star operator

$$\underset{\tau'_g}{\star} : \Lambda^p T^*M \to \Lambda^{4-p} T^*M$$
$$A_p \mapsto \underset{\tau'_g}{\star} A_p := \tilde{A}_p \tau'_g \tag{20}$$

Now, there are only two possibilities for τ'_g. Either $\tau'_g = \tau_g$ or $\tau'_g = -\tau_g$. In the second case we say that τ'_g defines a *negative* or *left handed orientation*. It is obvious that in this case we have

$$\underset{\tau'_g}{\star} A_p := - \underset{\tau_g}{\star} A_p, \tag{21}$$

but we insist: both $\underset{\tau'_g}{\star} A_p$ and $\underset{\tau_g}{\star} A_p$ are legitimate pair 4-forms.

Following Feynman, we take the view that only the field F is fundamental and that the charge carriers move in the vacuum (Lorentz vacuum) even when they are inside a medium, where they scatter from time to time the particles constituting the medium. It is then necessary, first of all to find the relation between G and F for the vacuum. It is an empirical fact that once a spacetime orientation τ_g is fixed (by arbitrary choice) we get a *correct* description of electromagnetic phenomena in vacuum[25] by taking

$$G := \underset{\tau_g}{\star} F \tag{22}$$

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[22] See, e.g., [7] for details.

[23] Of course, our statement is true only for someone that knows a little bit of Clifford algebra, as it is supposed to be the case of a reader of the present article.

[24] The Clifford product in this paper is represented by juxtaposition of symbols following the convention of [7].

[25] Of course, any formulation of electrodynamics in a medium must take as true the equations in vacuum and the properties of matter which are supposed to be described (due to the obvious difficulties with the many body problem) by an approximate phenomenological theory derived, e.g., from quantum mechanics. This is the point of view of Feynman [47] which we endorse.

Remark 5. Until now, we have only used pair forms in our formulation of electromagnetism, but we call the reader's attention to the fact that if we choose the opposite spacetime orientation τ_g' and $\underset{\tau_g'}{\star}$, we must put

$$G = -\underset{\tau_g'}{\star} F, \tag{23}$$

if we want to preserve the non homogeneous Maxwell equation $dG = -\mathbf{J}$.

4 Impair forms

The definition of the Hodge dual leaves it clear that different orientations (i.e., different *pair* volume element forms differing by a sign) produce duals – in the Hodge's sense – differing by a sign. This elementary fact is sometimes confused with the concept of impair forms introduced by de Rham [3]. From a historical point of view it must be recalled that de Rham pair and impair forms are only a modern reformulation of objects already introduced by Weyl [5] and then by Schouten [4].

Let $\{e_\mu\}$ and $\{e'_\mu\}$ be arbitrary bases for sections of $TU \subset TM$ and $TU' \subset TM$ ($U' \cap U \neq \varnothing$) and $\{\theta^\mu\}$ and $\{\theta'^\mu\}$ be respectively bases for $\sec \Lambda^1 T^*U \subset \sec \Lambda T^*M \hookrightarrow \sec \mathcal{C}\ell(M,g)$ and $\sec \Lambda^1 T^*U' \subset \sec \Lambda T^*M \hookrightarrow \sec \mathcal{C}\ell(M,g)$ which are respectively dual to the bases $\{e_\mu\}$ and $\{e'_\mu\}$. Let $\omega = \frac{1}{4!}\omega_{i_0 \ldots i_3} dx^{i_0} \wedge \cdots \wedge dx^{i_3}$ and $\omega' = \frac{1}{4!}\omega'_{i_0 \ldots i_3} dx'^{i_0} \wedge \cdots \wedge dx'^{i_3}$ and let τ_g be the orientation of the spacetime, which we recall is a *free* choice.

Definition 6. We say that the ordered coframe basis $\{\theta^\mu\}$, $\{\theta'^\mu\}$ (or simply ω, ω') are positive or right-handed oriented relative to τ_g if

$$o(\omega) := -\omega \cdot \tau_g > 0, \; o(\omega') = -\omega' \cdot \tau_g > 0, \tag{24}$$

and if

$$o(\omega) := -\omega \cdot \tau_g < 0, \; o(\omega') = -\omega' \cdot \tau_g < 0. \tag{25}$$

the bases are said to be negative or left-handed oriented.

Remark 7. It is very important not to confuse the concept of orientation of a coframe basis given by $o(\omega)$ with the spacetime orientation given by τ_g. But of course, the orientation of a coframe changes if it is referred to another volume element with different orientation.

Remark 8. Also, suppose that a given manifold M is non orientable. In this case we define the relative orientation of the basis $\{\theta^\mu\}$, $\{\theta'^\mu\}$ on $U \cap U'$ by saying that they have the same orientation if $\omega \cdot \omega' > 0$ and opposite orientation if $\omega \cdot \omega' < 0$. In the following the symbol $o(\omega)$ will be used according to *Definition* 6 if we are referring to an orientable manifold. In the eventual case where we referred to a non oriented manifold that even does not carry a metric field, $o(\omega')$ will mean the relative orientation of a given basis $\{\theta'^\mu\}$ in $U \cap U'$ relative to $\{\theta^\mu\}$, given by sign of Jacobian determinant $\det L / |\det L|$.

Definition 9. An impair p-form field $\overset{\triangle}{A}_p$ is an equivalence class of pairs $(\overset{\triangle}{A}{}^{\omega}_p, o(\omega))$, where $\overset{\triangle}{A}{}^{\omega}_p \in \sec \Lambda^p T^*M$ *a pair form* – called the representative of $\overset{\triangle}{A}_p$ on the basis $\{\theta^\mu\}$ – is given by

$$\overset{\triangle}{A}{}^{\omega}_p = o(\omega)\frac{1}{p!}\overset{\triangle}{A}_{i_1 \ldots i_p}\theta^{i_1} \wedge \cdots \wedge \theta^{i_p} \tag{26}$$

Given the pairs $(\overset{\triangle}{A}{}^{\omega}_p, o(\omega))$ and $(\overset{\triangle}{A}{}^{\omega'}_p, o(\omega'))$ where $\overset{\triangle}{A}{}^{\omega'}_p \in \sec \Lambda^p T^*M$ is given by

$$\overset{\triangle}{A}{}^{\omega'}_p = o(\omega')\frac{1}{p!}\overset{\triangle}{A'}_{j_1 \ldots i_p}\theta^{j_1} \wedge \cdots \wedge \theta^{j_p} \tag{27}$$

we say that they are equivalent if one of the two cases holds:

$$\begin{array}{lll} (a) & \text{if } o(\omega)=o(\omega'), & \overset{\triangle}{A}{}_p^{\omega} = \overset{\triangle}{A}{}_p^{\omega'} \\ (b) & \text{if } o(\omega)=-o(\omega'), & \overset{\triangle}{A}{}_p^{\omega} = -\overset{\triangle}{A}{}_p^{\omega'} \end{array} \tag{28}$$

Remark 10. Let $\{e_\mu = \frac{\partial}{\partial \mathrm{x}^\mu}\}$ and $\{e'_\mu = \frac{\partial}{\partial x'^\mu}\}$ be coordinate bases where $\{\mathrm{x}^\mu\}$ are global coordinates in the Einstein Lorentz-Poincaré gauge for $U \subset M$ and $\{x^\mu\}$ coordinates for $U' \subset M$. Now the orientation of $\{\gamma^\mu = d\mathrm{x}^\mu\}$ being taken as *positive*, if we *simply* write (as did de Rham)

$$\overset{\triangle}{A}{}_p^{\omega} = \frac{1}{p!}\overset{\blacktriangle}{A}_{i_1\ldots i_p} d\mathrm{x}^{i_1}\wedge\cdots\wedge d\mathrm{x}^{i_p}, \tag{29}$$

$$\overset{\triangle}{A}{}_p^{\omega'} = \frac{1}{p!}\overset{\blacktriangle}{A'}_{j_1\ldots j_p} dx'^{j_1}\wedge\cdots\wedge dx'^{j_p}, \tag{30}$$

then we must have

$$\overset{\blacktriangle}{A'}_{j_1\ldots j_p} = \frac{\det L}{|\det L|} L^{i_1}_{j_1}\ldots L^{i_p}_{j_p}\overset{\blacktriangle}{A}_{i_1..i_p}, \tag{31}$$

with $\det L = \det(\frac{\partial \mathrm{x}^i}{\partial x'^j})$. Eq. (31) is the definition of an *impair* form given by de Rham [3] (see also [4]).

Remark 11. It is very important to note that, since according to *Definition 6* an impair p-form is an equivalence class of pairs $(\overset{\triangle}{A}{}_p^{\omega}, o(\omega))$ where each $\overset{\triangle}{A}{}_p^{\omega}$ is a pair $p-$form and $o(\omega)$ denotes the basis orientation. Recall that if spacetime is oriented and we define $o(\omega)$ by Eqs. (24) and (25), then it depends on the spacetime orientation τ_g, and it follows that each pair p-form representative of an impair p-form depends also on the choice of the spacetime orientation. Indeed, if we change the spacetime orientation from τ_g to $\tau'_g = -\tau_g$ the orientation of the coframe $\{\theta^\mu\}$ changes to $o'(\omega) = -\omega\cdot\tau'_g = -o(\omega)$.

We denote the bundle of impair p-forms by $\Lambda^p_- T^*M$ and the exterior bundle $\Lambda_- T^*M = \oplus_{p=0}^4 \Lambda^p_- T^*M$. Let $\overset{\triangle}{A}_p \in \sec \Lambda^p_- T^*M$ denote that the impair p-form field $\overset{\triangle}{A}_p$ is a section of $\Lambda^p_- T^*M$

Remark 12. We can easily show that $\Lambda_- T^*M$ as defined above is isomorphic to $L(M)\otimes\Lambda T^*M$ (whose sections are line-bundle-valued multiforms on M). We write [40, 41]

$$\Lambda^p_- T^*M \simeq L(M)\otimes\Lambda T^*M, \tag{32}$$

where $L(M)$ is the so called orientation line bundle of M, a vector bundle with typical fiber $\mathbb{R}$ and where the transition functions are defined as follows. Let $\{(U_\alpha,\varphi_\alpha)\}$be a coordinate covering of M with transition functions given by $t_{\alpha\beta} = \varphi_\alpha\circ\varphi_\beta^{-1}$. Then, the transition functions of $L(M)$ are given by $J(t_{\alpha\beta})/\left|J(t_{\alpha\beta})\right|$, where $J(t_{\alpha\beta})$ means the Jacobian of matrix of the partial derivatives of $t_{\alpha\beta}$. Under the above conditions we can write (with the usual abuse of notation) for a given $\overset{\triangle}{A} \in \sec(L(M)\otimes\Lambda T^*M)$,

$$\overset{\triangle}{A} = e_{(\alpha)}\otimes A_{(\alpha)} = e_{(\beta)}\otimes A_{(\beta)} \tag{33}$$

This formula leaves it clear once again that to start any game with impair forms we must, once we choose a given chart $(U_\alpha,\varphi_\alpha)$, to give by *convention* an orientation $e_{(\alpha)}$ for it and next we must choose a pair form $A_{(\alpha)} = A$ or its negative, i.e., $A_{(\alpha)} = -A$ to build $\overset{\triangle}{A}$. This choice depends of course on the applications we have in mind.

4.1 The impair volume element

The impair 4-form $\overset{\triangle}{\tau}_g \in \sec \Lambda^4_- T^*M$ whose representative in an arbitrary basis $\{dx^\mu\}$ supposed positive is given by

$$\overset{\triangle}{\tau}_g = \frac{1}{4!}\overset{\blacktriangle}{\tau}_{i_o i_1 i_2 i_3} dx^{i_1} \wedge \cdots \wedge dx^{i_p} = \overset{\blacktriangle}{\tau}_{0123} dx^0 \wedge \cdots \wedge dx^3$$
$$= \sqrt{\left|\det(g_{ij})\right|}dx^0 \wedge \cdots \wedge dx^3, \tag{34}$$

is sometimes called (see, e.g., [48]) the *pseudo volume element*[26]. Now, the representative of this impair form in the basis $\{dx'^\mu\}$ is according to the definition just given

$$\overset{\triangle}{\tau}{}'_g = \frac{1}{4!}\overset{\blacktriangle}{\tau}{}'_{i_o i_1 i_2 i_3} dx'^{i_1} \wedge \cdots \wedge dx'^{i_p} = \overset{\blacktriangle}{\tau}{}'_{0123} dx'^0 \wedge \cdots \wedge dx'^3$$
$$= \sqrt{\left|\det(g'_{ij})\right|}dx'^0 \wedge \cdots \wedge dx'^3, \tag{35}$$

where we used that $\sqrt{\left|\det(g'_{ij})\right|} = |\det \Lambda| \sqrt{\left|\det(g_{ij})\right|}$ and Eq. (31), i.e.,

$$\overset{\blacktriangle}{\tau}{}'_{0123} = |\det L| \overset{\blacktriangle}{\tau}_{0123}. \tag{36}$$

Note that Eq. (36) is different from Eq. (15) which defines the transformation rule *for the components of a pair volume element.*

We recall also that given a chart (U, φ) of the atlas of M, the integral of an impair n-form $\overset{\triangle}{\tau}_g$ on a compact region $R \subset U \subset M$ is according to de Rham's definition (with $\mathbf{R} = \varphi(R)$) given by

$$\int_R \overset{\triangle}{\tau}_g := \int_{\mathbf{R}} \overset{\blacktriangle}{\tau}_{0123} dx^0 dx^1 dx^2 dx^3, \tag{37}$$

and as a result of Eq. (31) we have in the chart (U', φ'), $R \subset U'$ and with $\mathbf{R}' = \varphi'(R)$,

$$\int_R \overset{\triangle}{\tau}{}'_g = \int_{\mathbf{R}'} |\det L| \overset{\blacktriangle}{\tau}_{0123} dx'^0 dx'^1 dx'^2 dx'^3$$
$$= \int_{\mathbf{R}} \overset{\blacktriangle}{\tau}_{0123} dx^0 dx^1 dx^2 dx^3, \tag{38}$$

which corresponds to the classical formula for variables change in a multiple integration. Thus the integral of an impair n-form on a n-dimensional manifold is independent of the orientation of R. This is *not* the case if we try to integrate a pair n-form. We briefly recall de Rham's theory [3] of how to integrate pair and impair p-forms living on an n-dimensional manifold M.

Remark 13. Suppose we assign the natural orientation to a 'rectangle' $U^p \subset \mathbb{R}^p$ by $\tau = d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^p$ (where $\{\mathbf{x}^i\}$ are Cartesian coordinates for $\mathbb{R}^p$). It is now a classical result due to de Rham that it is always possible to integrate a pair p-form $\alpha \in \sec \Lambda^p T^*M$ over an *inner* oriented p-chain, i.e., a parametrized

[26] Here it becomes obvious what we said in Remark 12. We want of course, that the volume of a compact region $U \subset M$ be a positive number. This implies that we must choose as elements in the trivialization of $L(M) \otimes \Lambda T^*M$, $1 \otimes \sqrt{\left|\det(g_{ij})\right|}dx^0 \wedge \cdots \wedge dx^3$ or $(-1) \otimes (-\sqrt{\left|\det(g_{ij})\right|}dx^0 \wedge \cdots \wedge dx^3)$ which is simply written as $\overset{\triangle}{\tau}_g = \sqrt{\left|\det(g_{ij})\right|}dx^0 \wedge \cdots \wedge dx^3$.

submanifold $\psi : U^p \to M$ endowed with an inner orientation [3, 9, 11, 48] $\bar{o}(U^p)$. Indeed, if $\{u^i\}$ are arbitrary coordinates covering $\psi(U^p)$, we have by definition

$$\int_{(\psi(U^r),\, \bar{o}(U^r))} \alpha = \int_{(U^r,\, \bar{o}(U^r))} \psi^* \alpha$$
$$:= \bar{o}(U^r) \int_{(U^r,\, \bar{o}(U^r))} \psi^* \alpha \left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^r} \right) du^1 \dots du^r, \tag{39}$$

where $\bar{o}(U^r) = \operatorname{sign} \det(\frac{\partial u^i}{\partial \mathbf{x}^j})$. Of course, if we assign a different orientation $\bar{o}'(U^r) = -\bar{o}(U^r)$ to U^p we have

$$\int_{(\psi(U^r), -\bar{o}'(U^r))} \alpha = - \int_{(\psi(U^r),\, \bar{o}(U^r))} \alpha. \tag{40}$$

It is now opportune to bethink that any impair n-form is always integrable over any compact n-dimensional manifold M, be it orientable or not. However, it is *not* always possible to integrate an impair r-form $\overset{\triangle}{\alpha}$ on a n-dimensional manifold over a parametrized submanifold $\psi : U^r \to M$ unless ψ is an outer orientable map, i.e., if we can associate an orientation to M on $\psi(U^r)$. In general it may be not be possible to do that, and thus, we cannot integrate $\overset{\triangle}{\alpha}$ over an orientable p-dimensional submanifold $S \subset \psi(U^r)$, unless S is endowed with an *outer* or *transverse* orientation, i.e., if at any point of $x \in V$, $T_xM = T_xS \oplus N$ (with any $n \in N$ being transverse to S, i.e., $n \notin T_xS$) then each transversal N can be oriented continuously as a function of $x \in V$. Let $(x^1, \dots, x^n)$ be coordinates covering $U \subset M$ such that $S \cap U$ is defined by $x^\iota = f(x^1, \dots, x^r)$, $\iota = r+1, \dots, n$. Of, course the vector fields $\frac{\partial}{\partial x^\iota}$, $\iota = r+1, \dots, n$, defined in U, are transverse to S. Given an orientation for $S \cap U$, there always exists a set of vector fields $\{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ in $T(S \cap U)$ that are positively oriented there, which can be extended to all TU by trivially keeping their components constant when moving out of V. In this way an *outer* orientation can be defined in U by saying that $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \frac{\partial}{\partial x^{r+1}}, \dots, \frac{\partial}{\partial x^n}\}$ defines, let us say, the positive orientation on U. Then, if $\overset{\triangle}{\eta} \in \sec \Lambda^r T^* U$, the pullback form $i^* \overset{\triangle}{\eta}$ on $S \cap U$ (where of course $i : S \to M$ is the embedding of the submanifold S on M) is well defined, and can be integrated [48]. Finally, recall that the orientations defined by $o(\omega)$ and $\bar{o}(U^r)$ are obviously related[27], and we do not need further explanation.

Remark 14. It is essential to recall that $\Lambda_- T^* M$ is not *closed* under the operation of exterior multiplication and indeed to have a closed algebra (with that product) we need to take into account that the exterior multiplication of forms of the same parity is always a pair form and the exterior multiplication of forms of different parities is always an impair form. Also, the scalar product of forms of the same parity gives a pair 0-form and the scalar product of forms of different parities gives an impair 0-form. Moreover, the differential operator d preserves the parity of forms.

Remark 15. If we insist in using pair and impair forms for formulating the *differential* equations of motion of a given theory, e.g., in a formulation of electromagnetism in an oriented spacetime (something that at this point the reader must be convinced that it is not necessary at all, as it clear from the presentation given above) we need to introduce an impair Hodge star operator.

4.2 The impair Hodge star operator

Definition 16. Let $\overset{\triangle}{\tau}_g \in \sec \Lambda_-^4 T^* M$ be an impair volume form. The impair Hodge star operator is the map

$$\underset{\overset{\triangle}{\tau}_g}{\star} : \Lambda^p T^* M \to \Lambda_-^{4-p} T^* M, \tag{41}$$

[27] See, e.g., Sect. 4.2.6 of [7].

$$\underset{\overset{\triangle}{\tau}_g}{\star} : \Lambda^p_- \to \Lambda^{4-p} T^* M \tag{42}$$

such that for any $A_p \in \sec \Lambda^p T^* M$ and $\overset{\triangle}{B}_p \in \sec \Lambda^p_- T^* M$, we have

$$\begin{aligned} \underset{\overset{\triangle}{\tau}_g}{\star} A_p &:= \tilde{A}_p \overset{\triangle}{\tau}{}^{\omega}_g , \\ \underset{\overset{\triangle}{\tau}_g}{\star} \overset{\triangle}{B}_p &:= \tilde{\overset{\triangle}{B}}_p{}^{\omega} \overset{\triangle}{\tau}{}^{\omega}_g . \end{aligned} \tag{43}$$

Note that in Eq. (43) the Clifford product for the representatives of the impair forms is well defined (since according to Definition 9 each representative is an *even* form). Given the existence of impair and pair forms, many authors, e.g., [29–32, 48] advocate that even in an orientable manifold the formulation of electromagnetism must necessarily use besides the pair field strength $F \in \sec \Lambda^2 T^* M$ an impair exact 3-form $\overset{\triangle}{\mathbf{J}} \in \sec \Lambda^3_- T^* M$, which then defines the excitation field as an impair 2-form $\overset{\triangle}{G} \in \sec \Lambda^3_- T^* M$. We have thus for the vacuum situation

$$\begin{aligned} dF = 0,\ d\overset{\triangle}{G} &= -\overset{\triangle}{\mathbf{J}}, \\ \overset{\triangle}{G} &= \underset{\overset{\triangle}{\tau}_g}{\star} F. \end{aligned} \tag{44}$$

Remark 17. Authors [29–32, 48] (since the classical presentation of [37]) offers as the main argument for the necessity of using an impair $\overset{\triangle}{\mathbf{J}} \in \sec \Lambda^3_- T^* M$ in electromagnetism the following statement (which takes into account the *Remark* 13): "the value of the charge

$$Q = \int_S \overset{\triangle}{\mathbf{J}} \tag{45}$$

contained in a compact spacelike hypersurface $S \subset M$ must be independent of the (external) orientation of S"[28] and indeed, taking into account that $\overset{\triangle}{\mathbf{J}} = \underset{\overset{\triangle}{\tau}_g}{\star} J$, where J is given by Eq. (2) we must have

$$\int_S \overset{\triangle}{\mathbf{J}} = \sum\nolimits_i q^{(i)} \tag{46}$$

However, it is our view that the above argument is not a solid one. First, there is no empirical evidence that any spacelike surface $S \subset M$ where a real current is integrated does not possess an inner orientation that may be made consistent with the orientation of M. Thus, empirical evidence asserts that we may restrict ourself to only positively oriented charts (see also Sect. 4.3). But, even if we do not want to restrict

[28] Once we use impair forms, following [48] we may say that charge is a scalar. However, take care, in, e.g., [29] charge is said to be a pseudo-scalar. This apparent confusion comes out because in [29] it is discussed the properties of charge and of other electromagnetic quantities under an active parity operation and time reversal operators interpreted as appropriate mappings $\mathbf{p} : M \to M$ and $\mathbf{t} : M \to M$, where $M \simeq \mathbb{R}^4$ is the manifold entering in the structure of Minkowski spacetime. In that case we can show, e.g., that if $\overset{\triangle}{\mathbf{J}}$ is an *impair* 2-form than the pullback form $\overset{\triangle}{\mathbf{J}}{}' = \mathbf{p}^* \overset{\triangle}{\mathbf{J}} = -\overset{\triangle}{\mathbf{J}}$ and thus if $Q = \int \overset{\triangle}{\mathbf{J}}$ it follows that $\int \overset{\triangle}{\mathbf{J}}{}' = -Q$, and this according to [29] justifies calling charge a pseudo-scalar. As we see a confusion of tongues are also present in our subject. We are going to discuss more details of this particular issue in another publication.

ourselves to the use of positively oriented charts we must not forget that to perform the integral $\int_S \overset{\triangle}{\mathbf{J}}$ using a chart (U, φ), $S \subset U$ with coordinates $\{x^\mu\}$ we must choose an orientation recall Remark 13) for that chart and pick a specific choice for the pair form representing $\overset{\triangle}{\mathbf{J}}$. Suppose we make the convention that the orientation of $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is positive. At our disposal there are $\mathbf{J}$ and $-\mathbf{J}$. Which one to choose? The answer is obvious, we choose $\mathbf{J}$, for in this case we will have $\int_S \overset{\triangle}{\mathbf{J}} = \sum_i q^{(i)}$. This means that the attribution of the charge parameters to particles need in order to define the current depends on a *convention*, the one described above.

What happens if we represent the current by a pair form $\mathbf{J} \in \sec \Lambda^3 T^* M$? In this case the integral

$$\int_S \mathbf{J}, \tag{47}$$

does depend on the orientation of the chart used for its calculation. Suppose that as in the case of the integration of the impair form we use a chart (U, φ), $S \subset U$ with coordinates $\{x^\mu\}$. Which orientation should we give to $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$? The answer is obvious. We must choose an orientation (the positive one) such that

$$\int_S \mathbf{J} = \sum_i q^{(i)}. \tag{48}$$

So, in both cases (use of impair or pair current forms) to *start the evaluation process we need* to make a convention in order to fix the charge parameters of the charges that enter the definition of current such that Eq. (46) and Eq. (48) are true.

Now, what happens if someone using pair forms decides to calculate $\int_S \mathbf{J}$ using a chart with a different orientation than the one previously used to fix the charge parameters of the particles? The answer is that he will find that the value of the integral given by Eq. (47) will be now $-\sum_i q^{(i)}$? Is this a puzzle? Of course not, the value is negative because he is using a *different* convention than the previous one.

What we should ask is that if this break of convention changes the physics of electromagnetic phenomena? No, what happens is only that what was called a positive charge will now be called a negative charge and what was called a negative charge will now be called a positive one. No empirical fact will change, only some names. This is so because this change of names does not change any prediction of the theory concerning the motion of charged particles that besides the coupling parameter q also carries a coupling parameter m. We shall explicitly demonstrate this statement in Sect. 6 after we introduce the Clifford bundle formulation of electromagnetism. Here we emphasize again: the sign of a charge can be given meaning only by making it to interact with a charge that has been defined by convention as being positive (or negative)[29].

4.3 Teaching aliens what is right hand and what is left hand

The previous considerations of Remark 17 are valid only if the universe we live does not have regions composed of what we here called antimatter. Indeed, let us recall one of Feynman's stories (at page 103 of [49]) on the subject. Suppose we are in contact with some alien species, but only by the exchange, say of radio signals. Any intelligible communication needs a language and we suppose to build one doing something similar to the one proposed in the SETI program, starting with telling aliens what we mean by prime numbers and progressing to pictures, physics, and chemistry information. The concept of distance may be grasped by the aliens, e.g., by telling then how tall we are (in the mean), by expressing such number in mutually understood wavelengths of light. They can use that information to tell us how tall they are. We

[29] This point is well discussed in [38, 44] where the author uses an interesting homological approach in the formulation of the electromagnetic laws.

can also teach the aliens the concept of a man lifetime by expressing such number by the number of ticks of a light-frequency clock. To make agreement of some physical conventions and also to explain some social procedures among men (e.g., the fact that we shake hands when we meet, by extending our right hand) we need to explain them what is a right hand. How to do that?

As well known, until 1957 we could not answer that question. But, after the discover of the experiments showing parity violation in that year, we can explain to the aliens what the right hand is by asking them to repeat the original experiment done by Wu [50] et al with ^{60}Co, but in such a way that they must turn their apparatus (including the magnetic field generator in use) until the electrons come out in the downward direction, which we may define as the one of their local gravity pull. In such a situation the rotating nucleus will be with their spins up, i.e., rotating in the anti-clockwise direction as seen from the top. Before someone says that the aliens cannot see the ^{60}Co rotating we describe how we can teach them to amplify this anti-clockwise rotation (as seen from the top) in order that it becomes macroscopically visible. Indeed, all they need is to follow the following instructions. (A) Take a spherical conductor of radius a in electrostatic equilibrium with an uniform superficial charge density with total charge Q (i.e., charge as the ones carried by the atomic nucleus of the ^{60}Co) and which is magnetized with its dipole magnetic moment of modulus ς oriented in the same direction (the $\hat{\mathbf{z}}$-direction) of the magnetic moment of the ^{60}Co nucleus in his repetition of Wu's experiment. Such charged magnet has electromagnetic angular momentum stored in its electromagnetic field given by $\mathbf{L}_{\mathrm{em}} = \frac{2}{9}\varsigma Q a^2 \hat{z}$[30]. (B) The aliens are next instructed to discharge the magnet (suspended from the roof with an insulator) through the south pole. This makes the magnet to rotate anti-clockwise as seen from the top, in order to conserve the total angular momentum of field plus matter. Indeed a simple calculation [52, 53] shows that the mechanical angular momentum acquired by the sphere once completely discharged is[31] $\mathbf{L}_{\mathrm{mec}} = \frac{2}{9}\varsigma Q a^2 \hat{\mathbf{z}}$.

Of course, Feynman cautions us (page 107 of [49]) that after lots of communication if we finally can go into space and meet the aliens counterpart, if it happens that their leader extends its left hand to shake, stop immediately because that is proof that he is made of antimatter. This, of course, is because Wu's parity violation experiment constructed of antimatter would give the opposite result.

Feynman's story is important for the objectives of this paper because it shows that the charge argument is indeed a very week one. To fix the signal of the charge parameters that label particles and to describe their currents, we need to start with a convention, we need a local orientation, we need to know what a right hand is.[32]

4.4 Electromagnetism in a medium

Classical electromagnetism in a general medium is a very complicated subject since admitting with Feynman that the only the fundamental physical fields F and the current J generated by the particles carriers must enter the game are we immediately involved in an almost intractable many body system. However, it seems empirical fact that the equations

$$\begin{aligned} dF = 0,\ dG &= -\mathbf{J}, \\ G &= \chi(F) \end{aligned} \tag{49}$$

[30] This stored angular momentum in static electric plus magnetic field has been experimentally verified in [51].

[31] This result is obtained once we neglect the magnetic field associated with the discharging current and displacement current associated with the collapsing electric field, something justifiable if the current is small. If the current is not small some angular momentum will be carried by the radiation field, but of course at the end of the discharging process the sphere will be rotating.

[32] More recently Elitzur and Shinitzky [54] showed how to teach aliens what is right and what is left using the space asymmetry of molecules (L and D amino acids). However, to their method Feynman's caution also applies.

or

$$dF = 0,\ d\overset{\triangle}{G} = -\overset{\triangle}{\mathbf{J}},$$
$$\overset{\triangle}{G} = \overset{\triangle}{\chi}(F) \tag{50}$$

describes essentially all macroscopic electromagnetic phenomena on any medium contained in a world tube in $U \subset M$. In those equations κ and $\overset{\triangle}{\kappa}$ are multiform functions [55, 56] of the multiform variable F, i.e., for each $x \in U \subset M$ we have

$$\chi|_x : \Lambda^2 T_x^* M \to \Lambda^2 T_x^* M, \tag{51a}$$
$$\overset{\triangle}{\chi}\Big|_x : \Lambda^p T_x^* M \to \Lambda^2_- T_x^* M. \tag{51b}$$

Consider, e.g., Eq. (51b). The multiform function $\overset{\triangle}{\chi}$ is phenomenologically described, e.g., using coordinates in the *ELPG* by [14, 57]

$$\overset{\triangle}{G}{}^{\mu\nu} = \frac{1}{2}\left(\overset{\triangle}{\chi}{}^{\mu\nu\rho\sigma} F_{\rho\sigma} + \overset{\triangle}{\varsigma}{}^{\mu\nu\rho\sigma\iota\zeta} F_{\rho\sigma} F_{\iota\zeta} + \cdots\right). \tag{52}$$

A medium for which $\overset{\triangle}{\varsigma}{}^{\mu\nu\rho\sigma\iota\zeta} \neq 0$ is called nonlinear. For what follows we restrict our considerations only to linear media. In that case the constitutive multiform function $\overset{\triangle}{\chi}$ is a $(2, 2)$-extensor field [55, 56] and we have the decomposition [23]

$$\overset{\triangle}{\chi}{}^{\mu\nu\rho\sigma} = {}^{(1)}\overset{\triangle}{\chi}{}^{\mu\nu\rho\sigma} + {}^{(2)}\overset{\triangle}{\chi}{}^{\mu\nu\rho\sigma} + \overset{\triangle}{a}\,{}^{(3)}\varepsilon^{\mu\nu\rho\sigma}, \tag{53}$$

where ${}^{(1)}\overset{\triangle}{\chi}{}^{\mu\nu\rho\sigma}$ is a trace free symmetric part (with 20 independent components), ${}^{(2)}\overset{\triangle}{\chi}{}^{\mu\nu\rho\sigma}$ is the antisymmetric part (with 15 independent components) and $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol (with only 1 independent component). Finally, $\overset{\triangle}{a}$ is an impair 0-form field (also called a pseudo-scalar function) called the *axion* field. It has been a conjecture (called Post conjecture [30]) that $\overset{\triangle}{a}$ must be null for any medium.

However, recently it has been found that for Cr_2O_3, $\overset{\triangle}{a} \neq 0$. In [39] it is claimed that this fact even proves that we must use impair forms in the description of electromagnetism. However, those authors forget the following observation that can be found at page 22 of de Rham's book:

"Si la variété V est orientée, c' est-à-dire si elle est orientable et si l'"on a choisi une orientation ε, à toute forme impaire α est associée une forme paire $\varepsilon\alpha$. Par la suite, dans le cas d'une varieté orientable, en choissant une foi pour toutes une orientation, il serait possible d'éviter l'emploi des formes impaires. Mais pour les variétés non orientables, ce concept est réellement utile et naturel."

Now, for de Rham, an orientation ε is an impair 0-form field (i.e., an axion field) defined in the manifold M (that in his book is called V)[33]. So, all that the authors of [39] need to do in order to have only pair forms in their formulation of the electromagnetism of Cr_2O_3 is to multiply their impair objects by $\overset{\triangle}{a}$. So, contrary to popular believe the existence of an axion field does not imply that spacetime is non oriented. Quite the contrary is what is true.

[33] The reader can easily convince himself that this definition is equivalent to the one given in terms of an impair and a pair volume 4-forms $\overset{\triangle}{\tau}_g$ and τ_g. Indeed, take $\varepsilon = \overset{\triangle}{\tau}_g \cdot \tau_g$.

For some media where ${}^{(2)}\overset{\triangle}{\chi}{}^{\mu\nu\rho\sigma} = 0$, and $\overset{\triangle}{a} = 0$ we even find that the constitutive extensor may be described by [30]

$$\overset{\triangle}{\chi}{}^{\lambda\nu\sigma\kappa} = \sqrt{\det \mathbf{g}}(g^{\lambda\sigma}g^{\nu\kappa} - g^{\lambda\kappa}g^{\nu\sigma}) \tag{54}$$

where $g^{\lambda\sigma}g_{\lambda\nu} = \delta_\nu^\sigma$ and $g_{\mu\nu}$ are the components of an effective metric field $\mathbf{g} = g_{\mu\nu}d\mathrm{x}^\mu \otimes d\mathrm{x}^\nu$ for M. A particular medium with such characteristic is the vacuum in the presence of a gravitational field, but here we do not want to go deeply on this issue.

Remark 18. Until to this point the complete Minkowski spacetime structure $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ did not enter our formulation of electromagnetism. So, let us remark, first of all that from the point of view of an experimental physicist the parallelism rule defined by D is essential, since it is this parallel transport rule that permits him(her), e.g., to make to *parallel* filamentary currents and find their interaction behavior (as long ago did Ampere).

From a mathematical point of view, the connection D enter in our formulation of electromagnetism through the introduction of the Dirac operator acting on sections of the Clifford bundle $\mathcal{C}\ell(M, g)$.

Remark 19. Also, for media where the constitutive extensor can be put in the form given by Eq. (54) we can give an intrinsic presentation of electromagnetism using the Clifford bundle formalism by introducing an *effective* Lorentzian spacetime $(M, \mathbf{g}, \nabla, \tau_{\mathbf{g}}, \uparrow)$ where now, ∇ denotes the non-flat[34] Levi-Civita connection of $\mathbf{g}$, an effective Lorentzian metric determined by the constitutive tensor of that effective spacetime.

Remark 20. Before ending this section we have an important observation yet, concerning the metric free formulation of electromagnetism as presented, e.g., in [23]. There, it is admitted that M is an *oriented* connected, non compact, paracompact Hausdorff space. The authors say that a manifold with those characteristics always permits a codimension-1 foliation[35], a statement that is true [58]. However without any additional structure we cannot see how to foliate spacetime M as time $\times$ space ($\mathbb{R}\times S$), because we do not know a prior how to choose the dimension that represents time. In [23] the authors quickly introduce a global vector field $\mathbf{n}$ transverse to the folia, and the 3-dimensional manifold S of the foliation is defined by a manifold function $\sigma : M \to \mathbb{R}$ such that $\sigma(x) =$ constant and $\mathbf{n} \lrcorner d\sigma = 0$. It seems clear for us that $\mathbf{n}$ and $\Omega = d\sigma$ are nothing more than the universal vector field and the universal 1-form field defining the structure of absolute space and absolute time in the structure of Newtonian theory when that theory is formulated as a spacetime theory (for details, see [59]). Those observations can be translated in simple words: contrary to some claims only the bare structure of M is not enough for a formulation of electromagnetic theory as a physical theory.

5 The Clifford bundle formulation of electromagnetism

In a medium described by an effective Lorentzian spacetime $(M, \mathbf{g}, \nabla, \tau_{\mathbf{g}}, \uparrow)$ we may present the equations of electromagnetic theory as a single equation using the Clifford bundle $\mathcal{C}\ell(M, \mathrm{g})$ of *pair* differential forms. We recall [7] that in the Clifford bundle formalism the so called Dirac operator[36] $\boldsymbol{\partial}$ acting on sections of $\mathcal{C}\ell(M, \mathrm{g})$ is given by

$$\boldsymbol{\partial} = \boldsymbol{\partial}\wedge + \boldsymbol{\partial}\lrcorner \tag{55}$$

34 This term only means that the Riemann tensor $\mathbf{R}(\nabla) \neq 0$.

35 It admits also a 1-dimension foliation.

36 Please, do not confound the Dirac operator to the spin-Dirac operator which acts on sections of a spinor bundle. See details, e.g., in [7]. Here we recall that in an arbitrary basis $\{e_\mu\}$ for $TU \subset TM$, and $\{\theta^\mu\}$ for $T^*U \subset T^*M \subset \mathcal{C}\ell(M, \mathrm{g})$, the operator is given by $\boldsymbol{\partial} := \theta^\mu \nabla_{e_\mu}$. Take notice that $\mathrm{g} = g^{\mu\nu}e_\mu \otimes e_\nu$ if $\mathbf{g} = g_{\mu\nu}dx^\mu \otimes dx^\nu$, with $g^{\mu\nu}g_{\alpha\nu} = \delta_\alpha^\mu$.

It can be shown (see, e.g., [7]) that for a Levi-Civita connection we have $\boldsymbol{\partial}\wedge = d$ and $\boldsymbol{\partial}\lrcorner = -\delta$, where δ is the Hodge coderivative operator, such that for any $A_p \in \sec \Lambda^p T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ its action is given by:

$$\delta A_p = (-1)^p \underset{\tau_{\mathbf{g}}}{\star}^{-1} d \underset{\tau_{\mathbf{g}}}{\star} A_p. \tag{56}$$

We also have that:

$$\begin{aligned} -\delta A_p &= \partial \lrcorner A_p = \theta^\mu \lrcorner (\nabla_{e_\mu} A_p), \\ dA_p &= \partial \wedge A_p = \theta^\mu \wedge (\nabla_{e_\mu} A_p), \end{aligned} \tag{57}$$

and those expressions permit the simplification of many calculations. Recalling that $G = \underset{\tau_{\mathbf{g}}}{\star} F$ we get that $dG = -\mathbf{J}$ can be written defining $J = \underset{\tau_{\mathbf{g}}}{\star}^{-1} \mathbf{J} \in \sec \Lambda^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ as $\delta F = -J$. Indeed, we have,

$$\begin{aligned} d \underset{\tau_{\mathbf{g}}}{\star} F &= -\mathbf{J}, \\ \underset{\tau_{\mathbf{g}}}{\star}^{-1} d \underset{\tau_{\mathbf{g}}}{\star} F &= - \underset{\tau_{\mathbf{g}}}{\star}^{-1} \mathbf{J}, \\ \delta F &= -J \end{aligned}$$

Then, the two equations $dF = 0$ and $\delta F = -J$ can be *summed* if we suppose (as it is licit to do in the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$) that $F, G \in \sec \Lambda^2 T^{\,*}M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ and $\mathbf{J} \in \sec \Lambda^3 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ and we get *Maxwell equation*[37]

$$\boldsymbol{\partial} F = J. \tag{58}$$

Remark 21. We now show that Eq. (58) can also be obtained directly from the de Rham formulation of electromagnetism that uses pair and impair forms. Indeed, all that is need is to verify that the formula $d\overset{\triangle}{G} = -\overset{\triangle}{\mathbf{J}}$ in Eq. (44) – where $\overset{\triangle}{G} = \underset{\overset{\triangle}{\tau}_{\mathbf{g}}}{\star} F$ – can be written as $\delta F = -J$. Indeed, we have

$$\begin{aligned} d\overset{\triangle}{G} &= -\overset{\triangle}{\mathbf{J}}, \\ d \underset{\overset{\triangle}{\tau}_{\mathbf{g}}}{\star} F &= -\overset{\triangle}{\mathbf{J}}, \\ \underset{\overset{\triangle}{\tau}_{\mathbf{g}}}{\star}^{-1} d \underset{\overset{\triangle}{\tau}_{\mathbf{g}}}{\star} F &= - \underset{\overset{\triangle}{\tau}_{\mathbf{g}}}{\star}^{-1} \overset{\triangle}{\mathbf{J}}, \end{aligned}$$

and it is trivial to verify the formulas: $\underset{\overset{\triangle}{\tau}_{\mathbf{g}}}{\star}^{-1} d \underset{\overset{\triangle}{\tau}_{\mathbf{g}}}{\star} F = \delta F$ and $\underset{\overset{\triangle}{\tau}_{\mathbf{g}}}{\star}^{-1} \overset{\triangle}{\mathbf{J}} = \underset{\tau_{\mathbf{g}}}{\star}^{-1} \mathbf{J} = J \in \sec \Lambda^1 T^{\,*}M$. We conclude that the Clifford bundle formulation of electromagnetism given by Maxwell equation (Eq. (44)) is general enough to permit the two formulations of electromagnetism given above.

We are now ready to complete the formulation of electrodynamics as a physical theory. We restrict our presentation here in the case where the existence of the gravitational field must be ignored. As such our formulation will use the Minkowski spacetime structure introduced above and the Clifford bundle $\mathcal{C}\ell(M, g)$.

[37] No misprint here! Parodying Thirring [18] that said that the equations $dF = 0$ and $\delta F = 0$ were the 20^{th} century presentation of Maxwell equations, we say that the single equation $\boldsymbol{\partial} F = J$ is the 21^{th} century presentation of Maxwell equations.

6 The energy-momentum 1-forms of the electromagnetic and the matter fields

We start from Maxwell equation (with J the current of charged particles introduced by Eq. (2))

$$\boldsymbol{\partial} F = J \tag{59}$$

where in what follows, $\boldsymbol{\partial} = \theta^\alpha D_{e_\alpha} = \gamma^\mu \frac{\partial}{\partial \mathrm{x}^\mu}$ is the Dirac operator written with a general pair of dual basis $\{e_\alpha\}$ and $\{\theta^\alpha\}$ for $TU \subset TM$ and $T^*U \subset T^*M$ and with the basis $\{\frac{\partial}{\partial \mathrm{x}^\mu}\}$ and $\{\gamma^\mu = d\mathrm{x}^\mu\}$ for TM and $T^\star M$, with $\{\mathrm{x}^\mu\}$ coordinates in the Einstein-Lorentz-Poincaré gauge. Given Eq. (59) its reverse is

$$\tilde{F}\overleftarrow{\boldsymbol{\partial}} = J \tag{60}$$

where $\tilde{F}\overleftarrow{\boldsymbol{\partial}} := (D_{e_\alpha}\tilde{F})\theta^\alpha = (\frac{\partial}{\partial \mathrm{x}^\mu}\tilde{F})\gamma^\mu$. Multiplying Eq. (59) on the left by $\tilde{F}$ and Eq. (60) on the right by F and summing the resulting equations we get

$$\frac{1}{2}[\tilde{F}(\boldsymbol{\partial} F) + (\tilde{F}\overleftarrow{\boldsymbol{\partial}})F] = \frac{1}{2}(\tilde{F}J + JF), \tag{61}$$

Now, let $n = n^\alpha \gamma_\alpha \in \sec \Lambda^1 T^*M \hookrightarrow \mathcal{C}\ell(M,g)$ and $\boldsymbol{\partial}_n = \gamma^\alpha \frac{\partial}{\partial n^\alpha}$ acting on multiform functions of the multiform variable n. Consider moreover the extensor field [38] $T(n) = \frac{1}{2}\tilde{F}nF$. Now, observe that if we apply $\gamma^\alpha \cdot \boldsymbol{\partial}_n$ to the multiform function $\mathbf{f}(n) = \frac{\partial n}{\partial \mathrm{x}^\alpha}$ we get[39]

$$\begin{aligned}\gamma^\alpha \cdot \boldsymbol{\partial}_n \frac{\partial n}{\partial \mathrm{x}^\alpha} &= \eta^{\alpha\mu}\frac{\partial}{\partial n^\mu}(\frac{\partial}{\partial \mathrm{x}^\alpha} n^\beta \gamma_\beta) \\ &= \eta^{\alpha\mu}\frac{\partial}{\partial \mathrm{x}^\alpha}(\delta^\beta_\mu \gamma_\beta) = 0.\end{aligned} \tag{62}$$

Using Eq. (62) we can write the first member of Eq. (61) as

$$\begin{aligned}&\tilde{F}\gamma^\lambda \frac{\partial F}{\partial \mathrm{x}^\lambda} + \frac{\partial \tilde{F}}{\partial \mathrm{x}^\lambda}\gamma^\lambda F \\ &= \gamma^\lambda \cdot \boldsymbol{\partial}_n \left(\tilde{F}n\frac{\partial F}{\partial \mathrm{x}^\lambda} + \tilde{F}\frac{\partial n}{\partial \mathrm{x}^\lambda}F + \frac{\partial \tilde{F}}{\partial \mathrm{x}^\lambda}nF\right) \\ &= \gamma^\lambda \cdot \boldsymbol{\partial}_n \frac{\partial}{\partial \mathrm{x}^\lambda}(\tilde{F}nF) \\ &= \frac{\partial}{\partial \mathrm{x}^\lambda}(\tilde{F}\gamma^\lambda F)\end{aligned} \tag{63}$$

On the other hand the second member of Eq. (61) is just $-J\lrcorner F$. So, we have

$$\frac{\partial}{\partial \mathrm{x}^\alpha}T^\alpha = J\lrcorner F, \tag{64}$$

with the $T^\alpha \in \sec \Lambda^1 T^*M \hookrightarrow \mathcal{C}\ell(M,g)$ given by[40]

$$T^\alpha = \frac{1}{2}F\gamma^\alpha\tilde{F} \tag{65}$$

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38 An extensor field $T : \Lambda^1T^*M \to \Lambda^1T^*M$, $n \mapsto T(n)$ is a linear multiform function of the form field n.

39 See details on the derivation of multiform functions in [60].

40 An equation equivalent to Eq. (65) has been discovered by M. Riesz [61].

being the pair energy-momentum 1-forms of the electromagnetic field. Indeed, a simple calculation shows that

$$T^{\alpha\beta} = T^{\alpha} \cdot \gamma^{\beta} = \eta^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} \eta^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}, \tag{66}$$

a well known formula. Now, we contract Eq. (64) on the left with γ^{α} getting

$$\boldsymbol{\partial} \lrcorner T^{\alpha} = \gamma^{\alpha} \lrcorner (J \lrcorner F) \tag{67}$$

Now[41],

$$\begin{aligned} \gamma^{\alpha} \lrcorner (J \lrcorner F) &= (\gamma^{\alpha} \wedge J) \lrcorner F = -(\gamma^{\alpha} \wedge J) \cdot F \\ &= -F \cdot (\gamma^{\alpha} \wedge J) = -(\gamma^{\alpha} \lrcorner F) \cdot J, \end{aligned} \tag{68}$$

and Eq. (67) becomes (taking into account that $\boldsymbol{\partial} \lrcorner T^{\alpha} = -\delta T^{\alpha}$)

$$\delta T^{\alpha} = (\gamma^{\alpha} \lrcorner F) \cdot J \tag{69}$$

Defining $f^{\alpha} \in \sec \Lambda^4 T^* M$ as the *pair* density of force by

$$f^{\alpha} = [(\gamma^{\alpha} \lrcorner F) \cdot J] \tau_g, \tag{70}$$

where τ_g is a pair metric volume element, we obtain (the equivalent expressions)

$$\begin{aligned} f^{\alpha} &= [(\gamma^{\alpha} \lrcorner F) \cdot J)] \tau_g = \underset{\tau_g}{\star} [(\gamma^{\alpha} \lrcorner F) \lrcorner J] = (\gamma^{\alpha} \lrcorner F) \wedge \underset{\tau_g}{\star} J \\ &= (\gamma^{\alpha} \lrcorner F) \wedge \mathbf{J}, \end{aligned} \tag{71}$$

where, in particular, the last one is the pair density of force.

Remark 22. Note that we could return to Eq. (22) and get an impair density of force simply by replacing the pair volume element τ_g by an impair volume element $\overset{\triangle}{\tau}_g$, i.e., defining

$$\overset{\triangle}{f}{}^{\alpha} := [(\gamma^{\alpha} \lrcorner F) \cdot J] \, \overset{\triangle}{\tau}_g. \tag{72}$$

Such a formula was *postulated* in the presentation of electromagnetism in [23] However, as we just saw, that postulate for the coupling of the field F with the current $\mathbf{J}$ is not necessary in our approach, since that force density is already contained in Maxwell equation (Eq. (59)). We now write Eq. (69) as

$$d \underset{\tau_g}{\star} T^{\alpha} = (\gamma^{\alpha} \lrcorner F) \wedge \underset{\tau_g}{\star} J, \tag{73}$$

or

$$d \underset{\overset{\triangle}{\tau}_g}{\star} T^{\alpha} = (\gamma^{\alpha} \lrcorner F) \wedge \underset{\overset{\triangle}{\tau}_g}{\star} J \tag{74}$$

and both, of course, in components reads

$$\partial_{\nu} T^{\alpha\nu} = J_{\nu} F^{\nu\alpha} \tag{75}$$

[41] The sequence of identities in Eq. (68) may be found in Sect. 2.4.2 of [7].

6.1 Total energy-momentum conservation and the knockdown of the charge argument

Eq. (75) asserts that the energy momentum tensor of the electromagnetic field is not conserved. We expect that the *total* energy-momentum of the field and the charged particles is *conserved* since there is not a single experiment in Physics contradicting it, and so without much ado, recalling the definition of the *pair* energy-momentum 1-forms of the charged matter, the $\mathbf{T}^{\alpha}$ given by Eq. (7) we postulated that:

$$\delta T^{\alpha} + \delta \mathbf{T}^{\alpha} = 0, \tag{76}$$

which may be written as:

$$(\gamma^{\alpha} \lrcorner F) \wedge \underset{\tau_g}{\star} J = -d \underset{\tau_g}{\star} \mathbf{T}^{\alpha}. \tag{77}$$

or

$$(\gamma^{\alpha} \lrcorner F) \wedge \underset{\overset{\triangle}{\tau}_g}{\star} J = -d \underset{\overset{\triangle}{\tau}_g}{\star} \mathbf{T}^{\alpha}. \tag{78}$$

Eq. (77) is of course the *Lorentz force law*, and all objects in it are pair forms. On the other hand Eq. (78) is also an expression of the Lorentz force law and there are objects on it that are pair and others that are impair forms. Both equations give in our opinion the correct description of physical phenomena. However let us analyze here Eq. (77), since it permits us to knockdown again the *charge argument* [29–32, 48] which says that the density of current must be an impair 3-form, i.e., that we must use $\overset{\triangle}{\mathbf{J}} = \underset{\overset{\triangle}{\tau}_g}{\star} J \in \sec \Lambda^3 T^* M$ to calculate without ambiguity the charge in a certain spacelike region, say $S \in M$. To see this, recall that the total energy-momentum 1-form of matter in U at time $\mathrm{x}^0 = t$ in an inertial reference frame $\mathbf{I} = \frac{\partial}{\partial \mathrm{x}^0}$ is

$$P(t) = \left(\int_S \underset{\tau_g}{\star} \mathbf{T}^{\alpha} \right) \gamma_{\alpha}. \tag{79}$$

Now, if we change the orientation of S two things happen. What was called electric charge $q^{(i)} = \int_S \underset{\tau_g}{\star} J^{(i)} \Big|_{\sigma^{(i)}}$ of the i-particle changes into $-q^{(i)}$. F changes into $-F$ (despite the fact that it is a pair form) because of the formula used to calculate it (see Appendix). If we are interested in the motion of only a single small particle modeled by a thin world tube in an external field F , when integrating Eq. (79) we get that what we originally called energy at time t, $E^{(i)}(t) = \int_S \underset{\tau_g}{\star} \mathbf{T}^0 \Big|_{\sigma^{(i)}}$ of i-particle changes into $-E^{(i)}(t)$. This sign changes in Eq. (79) is compensated by the sign change that occurs in $\int_S (\gamma^{\alpha} \lrcorner F) \wedge \underset{\overset{\triangle}{\tau}_g}{\star} J_{(i)}$ and it follows that the prediction for the trajectory of that particle does not change if we change the orientation of S. And since trajectories of particles are all what are experimentally detected, it follows that the formulation of electrodynamics with only pair forms is compatible with the experimental facts[42].

Remark 23. If we take into account that $d \underset{\tau_g}{\star} F = - \underset{\tau_g}{\star} J$ we have that

$$Q = \int_{(S,o)} \underset{\tau_g}{\star} J = -\int_{(S,o)} d \underset{\tau_g}{\star} F = -\int_{(\partial S,o)} \underset{\tau_g}{\star} F \tag{80}$$

[42] We quote that our analysis of the sign to be atrtributed to charges and masses is in agreement to the one presented in [44], where (original in Italian) we can read (pp. 187): "... thus the electric charge is associated to volumes with external orientation and change sign when the external orientation is changed". Also, at pp. 201, we read: "...Thus the mass is associated to volumes with external orientation....and must change sign when the external orientation is changed". On this issue see also [45] (pp. 324).

where we have used Stokes theorem and where the orientations used for a spacelike hypersurface S $(\dim U = 3)$ and ∂S $(\dim \partial S = 2)$ are *internal* orientations. Now, as observed above if we change the internal orientation of U (without changing the orientation of M) the value of $Q = \int_{(S,o)} \underset{\tau_g}{\star} J$ changes sign i.e., we get $Q' = -Q = \int_{(S,-o)} \underset{\tau_g}{\star} J$. A change in the internal orientation of S (i.e., the internal orientation of TS) implies in a change of its external orientation since those orientations[43] are related by $(o_i(M) \equiv o_i(TM))$

$$o_i(TM) = o_e(TM/TS) \otimes o_i(TS), \tag{81}$$

where the external orientation of S is defined as the internal orientation of the quotient space TM/TS. Since $T\partial S$ is a subspace of TS we have that [62]

$$o_i(M) = o_e(TM/TS) \otimes o_i(TS) = o_e(TM/TS) \otimes o_e(TS/T\partial S) \otimes o_i(T\partial S),$$

which implies that the internal orientation $o_i(T\partial S)$ of $T\partial S$ does not change when we change the the internal orientation of U. This result is the one that warrants the consistence of the formalism since we already observed that when we change the orientation of U we need to change for consistence $F \mapsto -F$ and so $-\int_{(\partial S,o)} \underset{\tau_g}{\star}(-F) = -Q$.

7 The engineering formulation of electromagnetism without axial vector fields

We recall (see details in [7]) that for any $x \in M$, $\mathcal{C}\ell(T_x^*M, g_x) \simeq \mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, is the so called spacetime algebra. The even elements of $\mathbb{R}_{1,3}$ close a subalgebra called the Pauli algebra. That even subalgebra is denoted by $\mathbb{R}_{1,3}^0 \simeq \mathbb{R}_{3,0} \simeq \mathbb{C}(2)$. Also, $\mathbb{H}(2)$ is the algebra of the 2×2 quaternionic matrices and $\mathbb{C}(2)$ is the algebra of the 2×2 complex matrices. There is an isomorphism $\mathbb{R}_{1,3}^0 \simeq \mathbb{R}_{3,0}$ as the reader can easily convince himself. Choose a global orthonormal tetrad coframe $\{\gamma^\mu\}$, $\gamma^\mu = dx^\mu$, $\mu = 0,1,2,3$, and let $\{\gamma_\mu\}$ be the reciprocal tetrad of $\{\gamma^\mu\}$, i.e., $\gamma_\nu \cdot \gamma^\mu = \delta_\nu^\mu$. Now, put

$$\sigma_i = \gamma_i\gamma_0, \ \mathbf{i} = -\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^5. \tag{82}$$

Observe that $\mathbf{i}$ commutes with bivectors and thus *acts* like the imaginary unity $\mathrm{i} = \sqrt{-1}$ in the even subbundle $\mathcal{C}\ell^0(M,g) = \bigcup_{x\in M} \mathcal{C}\ell^0(T_x^*M, g_x) \hookrightarrow \mathcal{C}\ell(M, \mathrm{g})$, which may be called the *Pauli bundle*. Now, the electromagnetic field is represented in $\mathcal{C}\ell(M,g)$ by $F = \frac{1}{2}F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu \in \sec \Lambda^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,g)$ with

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{83}$$

where (E_1, E_2, E_3) and (B_1, B_2, B_3) are the *usual* Cartesian components of the electric and magnetic fields. Then, as it is easy to verify we can write

$$F = \mathbf{E} + \mathbf{i}\mathbf{B}, \tag{84}$$

with, $\mathbf{E} = \sum_{i=1}^3 E_i\sigma_i$, $\mathbf{B} = \sum_{i=1}^3 B_i\sigma_i$.

43 Recall that the space of internal orientations of a given manifold has only two elements $\{+1, -1\}$ but it has an algebraic structure where the product of orientations is defined in an obvious way. Besides that given to different manifolds we can form the tensor product of their orientation spaces to obtain another orientation space. This is how Eq. (81) is obtained.

Remark 24. Although $\mathbf{E}$ and $\mathbf{B}$ are 2-form fields in $\mathcal{C}\ell(M, \mathrm{g})$ they may be identified, once we fix an inertial reference frame (i.e., fix the γ^0 field) with time dependent Euclidean vector fields $\vec{E}, \vec{B}$ and thus we call them " Euclidean vector fields" in $\mathcal{C}\ell^0(M, \mathrm{g})$.

For the electric current density $J_e = \rho\gamma^0 + J^i\gamma_i$ we can write

$$\gamma_0 J_e = \rho - \mathbf{j} = \rho - J^i\sigma_i. \tag{85}$$

For the Dirac operator we have

$$\gamma_0 \boldsymbol{\partial} = \frac{\partial}{\partial x^0} + \sum_{i=1}^{3} \sigma_i \partial_i = \frac{\partial}{\partial t} + \nabla. \tag{86}$$

Multiplying both members of Eq. (58) on the left by γ_0 we obtain

$$\begin{aligned} \gamma_0 \boldsymbol{\partial} F &= \gamma_0 J, \\ (\frac{\partial}{\partial t} + \nabla)(\mathbf{E} + \mathbf{i}\mathbf{B}) &= \rho - \mathbf{j} \end{aligned} \tag{87}$$

From Eq. (87) we obtain

$$\partial_0 \mathbf{E} + \mathbf{i}\partial_0 \mathbf{B} + \nabla \bullet \mathbf{E} + \nabla \curlywedge \mathbf{E} + \mathbf{i}\nabla \bullet \mathbf{B} + \mathbf{i}\nabla \curlywedge \mathbf{B} = \rho - \mathbf{j}. \tag{88}$$

In Eq. (88) for any "vector field" $\mathbf{A} \in \mathcal{C}\ell^0(M, \mathrm{g})(\hookrightarrow \mathcal{C}\ell(M, \mathrm{g}))$,

$$\begin{aligned} \nabla \bullet \mathbf{A} &= \sigma_i \bullet (\partial_i \mathbf{A}), \\ \nabla \curlywedge \mathbf{A} &= \sigma_i \curlywedge (\partial_i \mathbf{A}), \end{aligned} \tag{89}$$

with the symbols $\bullet$ being defined through

$$\begin{aligned} \sigma_i \bullet \sigma_j &= \frac{1}{2}(\sigma_i\sigma_j + \sigma_j\sigma_i) = \delta_{ij}, \\ \sigma_i \curlywedge \sigma_j &= \frac{1}{2}(\sigma_i\sigma_j - \sigma_j\sigma_i). \end{aligned} \tag{90}$$

We define next the *vector* product of two "vector fields" $\mathbf{C} = \sum_{i=1}^{3} C_i\sigma_i$ and $\mathbf{D} = \sum_{i=1}^{3} D_i\sigma_i$ as the dual (see, e.g., [7]) of the "bivector field" $\mathbf{C} \curlywedge \mathbf{D}$ through the formula

$$\mathbf{C} \times \mathbf{D} = -\mathbf{i}(\mathbf{C} \curlywedge \mathbf{D}). \tag{91}$$

Finally, for any "vector field" $\mathbf{A} \in \mathcal{C}\ell^0(M, \mathrm{g})(\hookrightarrow \mathcal{C}\ell(M, \mathrm{g}))$ we define the *rotational operator* $\nabla\times$ by

$$\nabla \times \mathbf{A} = -\mathbf{i}(\nabla \curlywedge \mathbf{A}). \tag{92}$$

Using these concepts we obtain from Eq. (88) by equating terms with the same grades (in the Pauli subbundle)

$$\begin{array}{llll} \text{(a)} & \nabla \cdot \mathbf{E} = \rho, & \text{(b)} & \nabla \times \mathbf{B} - \partial_0 \mathbf{E} = \mathbf{j}, \\ \text{(c)} & \nabla \times \mathbf{E} + \partial_0 \mathbf{B} = 0, & \text{(d)} & \nabla \cdot \mathbf{B} = 0, \end{array} \tag{93}$$

which we recognize as the system of Maxwell equations in the usual vector (engineering) notation. However, the following remark is necessary.

Remark 25. From the above developments we see that a direct formulation of electromagnetism using time dependent fields, which are taken as sections of the Pauli subbundle $\mathcal{C}\ell^0(M, \mathrm{g})$, uses *only* vector fields, once an *orientation (*say $\mathbf{i}$) is fixed, thus dispensing the axial vector fields of the traditional Gibbs-Heaviside formulation and the more sophisticated formalism of tensors and tensor densities introduced by [37] and presented as a necessity by some other authors. Moreover, Eq. (91) leaves also clear that the definition of the vector product depends – in each inertial frame (i.e., when we fix field γ^0) – on the choice of an orientation in the affine Euclidean rest space S [42] of that frame. It implies that if we change the orientation of S, i.e., choose $-\mathbf{i}$ (instead of $\mathbf{i}$) in the definition of the vector product, we need to change $\mathbf{B} \mapsto -\mathbf{B}$, which means that the circulation of the magnetic field around a (very long) wire conducting current is conventional [48].

8 Conclusions

We showed that in any relativistic spacetime $(M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow)$, which is necessarily an *orientable* and *time orientable* manifold, electromagnetism can be coherently formulated using only *pair* form fields or *pair* and *impair* form fields, contrary to some claims appearing in the literature. The use of pair and impair form fields is necessary only if a non orientable manifold models our universe. However, a manifold of this kind cannot (according to a well known result [43]) represent the spacetime of our universe, where spinor fields live. Moreover we showed that using the Clifford bundle of (pair) forms we can give a formulation of electromagnetism that is compatible with those two formulations using only pair form fields or pair and impair form fields. Each one of those formulations depends only on a mathematical choice that does not seem to imply in any observable consequence.

An eventual objection to our formulation not discussed above, appeared in [63], which claims that the description of electromagnetism using the Clifford bundle formalism is not consistent if magnetic monopoles exist. Now, using that formalism the generalized Maxwell equations read

$$\boldsymbol{\partial} F = J - \underset{\tau_g}{\star} J_m, \tag{94}$$

where J is the pair electric current 1-form field and J_m is the pair magnetic current 1-form field, and the claim in [63] is that the Clifford bundle formalism implies that $J_m = 0$ Since this statement appears from time to time it is opportune to recall here that it has been proved wrong in [7, 64], since based on a misunderstanding that says that if the electric charges are scalars the magnetic charges must be pseudo-scalars. It also must be said that even if magnetic monopoles do not exist, Eq. (94) is important. The reason is the following. It can be shown that in the Clifford bundle formalism the standard Dirac equation describing, say the interaction of an electron field to the electromagnetic field, is represented by an equation called the the Dirac-Hestenes [65] equation which can be put in the form of Eq. (94). Indeed, the Dirac-Hestenes equation is

$$\boldsymbol{\partial}\psi\, \gamma^2\gamma^1 + m\psi\gamma^0 + qA\psi = 0, \tag{95}$$

where ψ is a Dirac-Hestenes spinor field [7, 66, 67], a mathematical object represented in a given inertial frame $\mathbf{I} = \partial/\partial \mathrm{x}^0$ and once a spin-frame is fixed by a non homogeneous even section of the Clifford bundle[44],

$$\psi = S + F + \tau_g P \in \sec(\Lambda^0 T^* M + \Lambda^2 T^* M + \Lambda^4 T^* M). \tag{96}$$

It can then easily be shown substituting Eq. (96) in Eq. (95) that the Dirac-Hestenes equation can be written in the form of Eq. (94), where the "electric" and "magnetic" like currents are non linear functionals depending on S, F and P [7, 64].

[44] More precisely, Dirac-Hestenes spinor fields are some equivalent classes of even *non homogeneous* differential forms. See, [66, 67] for details.

Two observations are yet necessary. The first one is that we are sure that an attentive reader which has not been yet introduced to the Clifford bundle formalism may have become intrigued with our statement in the abstract that pair forms may be used coherently besides in electromagnetism, also in any other physical theory. We just mentioned that Dirac equation can be represented by sum of nonhomogeneous even sections of the Clifford bundle. But, someone may still eventually ask: and what about Einstein's gravitational theory which is formulated with a *symmetric* metric field as its fundamental field? Well, gravitational theory may also be formulated that field being represented by a set of four linearly independent 1-form fields living on M.The details of how this is done can be found in, e.g., [7, 68, 69] and that fact seems to give even more importance to the modern theory of differential forms which started with Cartan.

The second and last observation has to do with a possible question that may come to mind to some reader which knows discrete electrodynamics[45]. Indeed, that theory, as formulated, e.g., in [38, 44, 70, 72] with chains and cochains (as introduced in algebraic topology[46]) representing the observables uses two types of orientations, internal and external, associated to two different classes of chains and cochains (pair and impair). The introduction of those objects makes almost geometrically obvious the association of some physical quantities to one or another class. So, is it the case that in discrete electrodynamics we finally need to use pair and impair objects? Well, once again we have as answer that if the spacetime where the chains and cochains of discrete electrodymanics leave is *oriented*, we can again (if due care is taken) associate to each external oriented chain (cochain) an internal oriented chain (cochain) using a prescription analogous to the one given by Eq. (81) and then work only with internal oriented objects. We loose in this case the obvious geometrical meaning of some of the quantities, but the algebraic computations result correct if sue care is taken. This is analogous to de Rham's statement quoted above that in an orientable manifold (once an orientation is chosen) to each impair form there corresponds a unique pair form.

Note added in proof

Authors of [74] said that their comment it is a reaction to our paper which is related to classical electrodynamics. It is our opinion that we deal appropriately with all their comments (some of them, unfortunately non appropriate), but here for the benefit of our readers it turns out necessary to repeat a *crucial* remark of our paper and make (due to space limitations) only some few comments. The first and more important is that what we show in our paper that in an *oriented* Lorentzian spacetime we can formulate classical electrodynamics using only *pair* form fields, viewed as sections of an appropriate Clifford bundle (thus dispensing the use of impair form fields) in a coherent way using good mathematics. This is due to the fact quoted in our paper and first spelled by de Rham (our reference [3]):

> ≫Si la variété V est orienté, c'est-à-dire si elle est orientable et si l'on a choisi une orientation ε, à toute forme impaire α est associée une forme paire $\varepsilon\alpha$. Par la suite, dans le cas d'une varieté orientable, en choissant une foi pour toutes une orientation, il serai possible d'éviter l'emploi des formes impaires. Mais pour les variétés non orientables, ce concept est réellement utile et naturel.≪

Using only pair forms, of course, does not mean – contrary to what our critics think and spell – that the resulting differential equations of our theory are *not* invariant under arbitrary coordinate transformations. This is so because all differential equations in our approach are written intrinsically. However, when using pair forms the sign of a charge resulting from the evaluation of the integral of a *pair* current 3-form $\mathbf{J}$ depends of course, on the handiness of the coordinate chart using for performing the evaluation. The

[45] Discrete Electrodynamics is part of the so called Discrete Physics formulated in a basic affine manifold and where the main idea is to dispense the fields and the differential equations of the standard physical theories and work with (set) functions (the cochains) whose domain is the space of chains and the whose range is some additive group, which must be chosen for each particular theory. The interested reader should consult: http://discretephysics.dic.units.it/

[46] See, specially Chapter 7 of [71].

relevant question is: *does it imply any contradiction with observed phenomena*? As clearly shown in our paper through a very careful analysis using good mathematics the answer is no.

However, our critics are not happy with our analysis and continue to insist *ad nauseam* that charge does not have a screw sense and as such the electromagnetic current must be an impair (twisted) **3**-form field **J** because they "may want to put charge on a (non-orientable) Möbius strip ...". Well, suppose for a while that the Möbius strip $M\ddot{o}$ is sitting on $\mathbb{R}^3$ (the rest space of an inertial frame). To eventually calculate its charge we need to start with a 2-form surface charge density $\mathcal{J}$ defined on $\mathbb{R}^3$. Now had our critics read our Remark 13 they could be recalled of the fact that being $\mathcal{J}$ a pair *or* an impair 2-form we cannot define its integral over the Möbius strip. So, we conclude that only in fiction can someone think in putting a real physical charge distribution (made of elementary charge carriers) on a Möbius strip (it should sit on $M\ddot{o} \times \mathbb{R}$), and leaving this physical impossibility aside we cannot see any necessity for the use of impair forms.

Moreover, our critics said that our statement that the Clifford bundle works only with pair forms and could not apply to Physics if there is any real need for the use of impair forms is *unsubstantiated* and justify their assertion quoting preliminary algebraic studies by Demers (their reference [33]) which deal with a *non-associative* 'Clifford like' algebra structure involving pair and impair forms. That structure has *nothing* to do with the Clifford algebra used (as fibers) in our Clifford bundle, which is an *associative* algebra, a property that makes that formalism a very powerful computational tool. We recall also that as detailed in our paper our formalism which writes 'Maxwell equation' (no misprint here) with *pair* differential forms can be split in *two* different ways. The first one results in two equations using only pair forms and the second one (after introducing an impair Hodge star operator) results in an equation using pair forms and another one using impair forms.

However to do that it is crucial to understand that there exists two different Hodge star operators, one *pair* and one *impair*. They are very distinct objects, often confused (as we explained in detail in our paper). We recall that to have that fact in mind is important because without the *explicit* introduction of the impair Hodge dual operator the claim of our critics (that do not even mention that object) that Maxwell equation in the Clifford bundle splits in an equation for a pair form and one involving impair forms is simply meaningless and indeed the calculation they present (the correct ones dealing with this issue is in our paper) results in a set of two equations involving only pair forms, contrary to their claim. Our critics said that statement that we get from Maxwell equation the Lorentz force law is *empty* because we did not define what is F. Well, this is simply not true.

In our approach it is clear that F is taken as a physical field represented by a 2-form field living in Minkowski spacetime and satisfying Maxwell equation, where a current 1-form J (formed from the charged matter carriers) acts as source of F. We next find the energy-momentum 1-form fields of F ($T^{\alpha} = -1/2F\gamma^{\alpha}F$) and impose that the total energy-momentum tensor of the F field plus the charged matter field is conserved. Under those well defined conditions we proved that the coupling of F with J must be given by the Lorentz force law, which must then be used in the *operational* way in which those objects must be used when one is doing Physics. It is in this *sense* that we said that such law need not be postulated in classical electrodynamics, and we are sure that any attentive reader of our paper will understand what we said and what we proved.

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A How to calculate F

A.1 Green's identity for differential forms

In this section M is a n-dimensional differentiable manifold and $\mathbf{g} \in \sec T_0^2 M$ is a metric on M of arbitrary signature (p,q), with $p+q=n$. Moreover, we denote by $\mathtt{g} \in \sec T_2^0 M$ the metric on the cotangent bundle such that in an arbitrary coordinate basis where $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$ and $\mathtt{g} = g^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}$, it is $g_{\mu\nu} g^{\mu\alpha} = \delta_\nu^\alpha$. We suppose moreover that $\Lambda T^* M$ and $\mathcal{C}\ell(M,\mathtt{g})$ are respectively the exterior and Clifford algebra bundles of M. Let $P \in \sec \Lambda^p T^* M \subset \sec \mathcal{C}\ell(M,\mathtt{g})$. We shall derive an integral identity involving P, dP a δP and a Green (*extensor*) distribution[47] $\mathbf{G}_{\breve{x}} \in \sec \Lambda^p T^* \breve{M} \otimes \sec \Lambda^{n-p} T^* M$ that is a generalization of the well known Green's identities of classical vector calculus. This identity is crucial in order to obtain a formula solving certain differential equations satisfied by P.

Let $\{\theta^j, \theta_j\}$ be a pair reciprocal bases for $\Lambda^1 T^* M \hookrightarrow \mathcal{C}\ell(M,\mathtt{g})$. In what follows the notation $\breve{\theta}_i$ $(\breve{\theta}^i)$ means that these forms are calculated at a point $\breve{x} \in \breve{M}$. Now, we introduce the Dirac extensor distribution $\delta_{\breve{x}} \in \sec \Lambda^p T^* \breve{M} \otimes \sec \Lambda^{n-p} T^* M$ by

$$\int \delta_{\breve{x}} \wedge P(x) = P(\breve{x}). \tag{97}$$

where $\delta_{\breve{x}}$ has support only at $\breve{x}$. If $\{x^i\}$ are the coordinate of a chart of an atlas of M and if we choose $\{\theta^j = dx^j, \theta_j = g_{ij} dx^i\}$ then we can easily verify that

$$\begin{aligned} \delta_{\breve{x}} &= \frac{(-1)^{p(n-p)}}{p!} \breve{\theta}_{i_1 \ldots i_p} \otimes \star\theta^{i_1 \ldots i_p} \delta(x-\breve{x}), \\ \delta(x-\bar{x}) &= \delta(x^1 - \breve{x}^1) \ldots \delta(x^n - \breve{x}^n), \\ \breve{\theta}_{i_1 \ldots i_p} &= \breve{\theta}_{i_1} \wedge \cdots \wedge \breve{\theta}_{i_p},\ \theta^{i_1 \ldots i_p} = \theta^{i_1} \wedge \cdots \wedge \theta^{i_p}, \end{aligned} \tag{98}$$

where in Eq. (98) $\delta(x^i - \breve{x}^i)$, $i = 1, 2, \ldots, n$ are the usual (scalar) Dirac measures.

The Green distribution is supposed to satisfy the following differential equation

$$\Box \mathbf{G}_{\breve{x}} = -(d\delta + \delta d)\mathbf{G}_{\breve{x}} = \delta_{\breve{x}}. \tag{99}$$

We now prove the following identity:

$$\begin{aligned} \delta_{\breve{x}} \wedge P =& (-1)^{n+p}[d\mathbf{G}_{\breve{x}} \wedge \delta P - \delta \mathbf{G}_{\breve{x}} \wedge dP] \\ &- d[\delta \mathbf{G}_{\breve{x}} \wedge P - (-1)^{np+p+s+1} \underset{\tau_g}{\star} P \wedge \underset{\tau_g}{\star} d\mathbf{G}_{\breve{x}}]. \end{aligned} \tag{100}$$

We start with the product $d\mathbf{G}_{\breve{x}} \wedge \delta P$ and make some transformations on it using the definition of the Hodge coderivative and some other well known identities involving the exterior product[48]. We then have

$$\begin{aligned} d\mathbf{G}_{\breve{x}} \wedge \delta P =& (-1)^{n(p+1)+s+1} d\mathbf{G}_{\breve{x}} \wedge \underset{\tau_g}{\star} d \underset{\tau_g}{\star} P \\ =& (-1)^{np+n+s+1} d \underset{\tau_g}{\star} P \wedge \underset{\tau_g}{\star} d\mathbf{G}_{\bar{x}} \end{aligned}$$

---

[47] The distribution $\mathbf{G}_{\breve{x}}$ is also called a p-form-valued de Rham current. Rigorously we should write $P \in \sec \mathfrak{D}'(M, \Lambda^p T^* M) \subset \sec \mathfrak{D}'(M, \mathcal{C}\ell(M,\mathtt{g}))$ and $\mathbf{G}_{\breve{x}} \in \sec \Lambda^p T^* \breve{M} \otimes \sec \mathfrak{D}'(M, \Lambda^{n-p} T^* M) \hookrightarrow \sec \bigwedge^p T^* \bar{M} \otimes \sec D'(M, C\ell(M,\mathtt{g}))$ where $\sec \mathfrak{D}'(M, \Lambda^{n-p} T^* M)$ is the space of the linear functionals over the sections of $\Lambda^p T^* M$ of p-forms of compact support (in the sense of its action as, e.g., in Eq. (97). $\breve{M}$ is a copy of M and is there to recall that $\mathbf{G}_{\breve{x}}$ is a two point distribution.

[48] See, e.g., Sect. 2.4.2 of [7].

$$
\begin{aligned}
&=(-1)^{s+1}d(\underset{\tau_g}{\star}P\wedge\underset{\tau_g}{\star}d\mathbf{G}_{\breve{x}})-(-1)^{n+p}\delta d\mathbf{G}_{\breve{x}}\wedge P\\
&=(-1)^{s+1}d(\underset{\tau_g}{\star}P\wedge\underset{\tau_g}{\star}d\mathbf{G}_{\breve{x}})+(-1)^{n+p}[(-\delta d-d\delta)\mathbf{G}_{\breve{x}}\wedge P]\\
&\quad+(-1)^{n+p}d\delta\mathbf{G}_{\breve{x}}\wedge P\\
&=(-1)^{s+1}d(\underset{\tau_g}{\star}P\wedge\underset{\tau_g}{\star}d\mathbf{G}_{\breve{x}})+(-1)^{n+p}\delta_{\breve{x}}\wedge P+(-1)^{n+p}d(\delta\mathbf{G}_{\breve{x}}\wedge P)\\
&\quad+\delta\mathbf{G}_{\breve{x}}\wedge dP,
\end{aligned}
$$

from where Eq. (100) follows.

Integrating both sides on the n-dimensional region $\mathcal{U}\subset M$ we have[49]

$$
\begin{aligned}
P(\breve{x})=&(-1)^{n+p}\int_{\mathcal{U}}[d\mathbf{G}_{\breve{x}}\wedge\delta P-\delta\mathbf{G}_{\breve{x}}\wedge dP]\\
&-\int_{\partial\mathcal{U}}\delta\mathbf{G}_{\breve{x}}\wedge P-(-1)^{n+p+s}\underset{\tau_g}{\star}d\mathbf{G}_{\breve{x}}\wedge\underset{\tau_g}{\star}P].
\end{aligned}
\tag{101}
$$

A.2 Solution of $\boldsymbol{\partial}F=J$

We now applies the above formula for solving the equation $\boldsymbol{\partial}F=J$ in a Minkowski manifold. We start by choosing a chart with coordinates $\{\mathrm{x}^\mu\}$ in the *ELPG. We write as in the text* $\gamma^\mu=d\mathrm{x}^\mu$ and $\gamma_\mu=\eta_{\mu\nu}\gamma^\nu$. Then, since $\boldsymbol{\partial}F=J$ is equivalent to $dF=0$ and $\delta F=-J$, we have using the retarded solution

$$
\begin{aligned}
&\mathbf{G}_s(\mathrm{x}-\breve{\mathrm{x}})=\frac{1}{2}\breve{\gamma}_{\mu_1\mu_2}\otimes\underset{\tau_g}{\star}\gamma^{\mu_1\mu_2}G_s(\mathrm{x}-\breve{\mathrm{x}}),\\
&\boldsymbol{\partial}^2G_s(\mathrm{x}-\breve{\mathrm{x}})=\delta(\mathrm{x}-\breve{\mathrm{x}})
\end{aligned}
\tag{102}
$$

where G_s is the scalar Green *retarded* function (see, e.g., [73]), which vanishes outside the light cone at x. Then

$$
F(\mathrm{x})=-\int_{\mathcal{U}}d\mathbf{G}_s(\mathrm{x}-\breve{\mathrm{x}})\wedge J(\breve{\mathrm{x}})-\int_{\partial\mathcal{U}}\delta\mathbf{G}_s(\mathrm{x}-\breve{\mathrm{x}})\wedge F(\breve{\mathrm{x}})+\underset{\tau_g}{\star}d\mathbf{G}_s(\mathrm{x}-\breve{\mathrm{x}})\wedge\underset{\tau_g}{\star}F(\breve{\mathrm{x}}),\tag{103}
$$

and supposing that F vanishes on the boundary $\partial\mathcal{U}$ we end with

$$
F(\mathrm{x})=-\int_{\mathcal{U}}d\mathbf{G}(\mathrm{x}-\breve{\mathrm{x}})\wedge J(\breve{\mathrm{x}}).\tag{104}
$$

This equation shows explicitly that $F\rightarrow-F$ when we decide to *relabel* the charges entering J from $q^{(i)}$ to $-q^{(i)}$ something that as already discussed in the text happens if we calculate $\int\underset{\tau_g}{\star}J$ in a chart with a different orientation than the positive one defined by $\gamma^0\wedge\gamma^1\wedge\gamma^2\wedge\gamma^3$.

[49] Analogous equation to Eq. (101) appears in Thirring's book [18]. However take care on comparing the equations there and here because of some (-1) signs arising due to different definitions of the Hodge coderivative.

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