Torsion Fields, the Extended Photon, Quantum Jumps, The Eikonal Equation, The Twistor Geometry of Cognitive Space and the Laws of Thought

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Abstract: A geometrical origin of quantum jumps in terms of torsion fields and the propagation of wave-front singularities given by the eikonal equation of geometrical optics, which lies at the basis of Fock's theory of gravitation, is introduced. A discussion on the connection between quantum jumps and a global time and space coordinates system is presented. The most general form of the solutions of the eikonal and wave equations in a quaternionic setting to obtain the representation of the photon as an extended singularity is formalized, as well as their twistor representations. Matrix logic and its connections to quantum field operators and hypernumbers are elaborated. The torsion geometry of matrix logic and the relations with quantum mechanical observables and quantum superposition, namely: the so called Schroedinger cat problem, the multivalued character of matrix logic and non-orientable surfaces - the Moebius band and the Klein bottle-, are presented. The plenum zero operator (defined by the matrix with all entries equal to 0), of matrix logic as a logical-quantum ground-state observable (which we shall call the *mind apeiron*) and its twistor eigenstates are introduced. The relation between the twistor representations of the quaternionic eikonal equation and those of the mind apeiron is discussed, establishing thus a relation between the extended structure of the photon and the eigenstates of the mind apeiron. This gives in principle a solution to the so-called mind-matter problem, surmounting Cartesian duality. We present a connection between the quaternionic structure in matrix logic and some metrics in cosmological models .

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1 Introduction.

In this article we shall deal with several issues which in principle might seem unconnected but are all related to apeiron. These issues are the space-time self-referential geometries with torsion [13] and the self-referential character of the photon -both terms to be explained below-. We shall treat them as the quaternionic solutions of the system given by the eikonal and wave equations, and the relation with the laws of thought. These laws will be addressed here, in terms of a multivalued logic given by matrix logic, an operator extension of the usual Aristotelian-Boolean connective two-valued logic. This matrix logic extends quantum, fuzzy, modal and Aristotelian-Boolean logics, showing that there is a close connection between quantum fields, logical operators and the torsion geometry of cognition. The latter appears as the coefficients in the commutator of the TRUE and FALSE operators, which turns out to be non-trivial due to their non-hermiticity and consequent non-duality. [2]

[2] We here introduce, for a sequential completeness of the presentation of this article, some elements we shall later retake. Logical operators are the representation of their truth values tables by 2×2 matrices acting on cognitive states given by Dirac-like bras or kets of the form $|q>= \begin{pmatrix} \bar{q} \\ q \end{pmatrix}$ with q an arbitrary real number (instead of being 0 or 1, as in Boolean logic) and $\bar{q} = 1 - q$ is the negation of q as in Boolean logic. Then, the definitions are: $\mathrm{TRUE}|q>=|0>$ and $\mathrm{FALSE}|q>=|1>$, with $|0>$ and $|1>$ the false and true *vectors*, respectively. The matrix representations -which we shall present further below- of these operators are non-self adjoint. In distinction with usual (hermitean) quantum observables, logical operators are generally non-hermitean although they have representations as quantum field operators and the reciprocal is also the case. A consequence of this non-hermiticity is that in contrast with the trivial duality of the true and false scalars of connective logic, 1 and 0 respectively (which is represented by the relations $\bar{1} = 0$ and $\bar{0} = 1$), by defining the *complement* $\bar{L}$ of a logical operator L, by $I - L$, where I is the identity operator, we obtain that $\overline{\mathrm{TRUE}} \neq \mathrm{FALSE}$ and $\overline{\mathrm{FALSE}} \neq \mathrm{TRUE}$. So, this notion of complementarity when restricted to scalar fields coincides with the dual operation of Boolean logic transforming conjuction into disjunction. This duality affirms the principle of non-contradiction of Aristotelian-Boolean logic: given any proposition, p, then p *and* not p is false, and thus the previous result proves the non-duality of TRUE and FALSE. Another important consequence of this non-hermiticity, is the appearance in matrix logic of a *logical* momentum operator and its relation to the Bohr-Sommerfeld quantization condition, as we shall present below. Finally, Lie symmetry groups (say of finite dimension n) have a canonical geometrical structure with non-null torsion which characterizes precisely the infinitesimal symmetries of their Lie algebras. Indeed, if we consider the structure coefficients C^a_{bc} of the Lie algebra defined by the commutator $[\xi_b, \xi_c] = C^a_{bc}\xi_a$ (summation convention of repeated indices whenever it applies), for any elements ξ_i (with $i = 1, \ldots, n$) of the Lie algebra, then the torsion of this canonical geometry of the Lie groups is given by $-C^a_{bc}$ [61]. In matrix logic FALSE and TRUE play the role of infinitesimal vectors (i.e. as vector fields, or linear operators) acting on the vector space of bras and kets under a superposition principle which we shall characterize below. We shall

The subject being a singularity (an irreductible form which is also a process) cognizes the world and simultaneously establishes himself as a self-aware observer through cognition and perception stemming from distinctions. These distinctions are primeval in being differences that make a difference in the sense introduced in [45]. These are distinctions which on being perceived, cognized, abstracted or interpreted, generate higher-order differences which amount to the universe of all manifestations, either virtual, processual, operational, algorithmic, formal, conceptual or real [47]; for further developments of an epistemology that departs from this notion of primeval distiction see [46]. Without distinctions in its manifold manifestations, the world would be homogeneous and imperceptible [13]. To introduce this conception we shall take the somewhat paradoxical approach of presenting it through a seemingly realistic approach, based on a geometrical theory for the characterization of quantum jumps. The latter will be characterized in terms of spacetime singularities produced by a torsion field. This field is the logarithmic differential of a wave function propagating on spacetime as a light-like singularity described by the eikonal equation of geometrical optics for light rays [11]. Yet, this realism follows from the peculiar embodiment of the fusion of object with subject that the absorbed photon is. Indeed, the absorbed photon is not an 'objective' structure, but rather a structure and a process constituted by its perception by the subject, and thus of second order. As stated simply [77], "...light is not seen; it is seeing", which in this article we shall prove to admit an extension: Light is seeing-thinking. Thus the photon is a difference, a quantum of action, which generates higher-order differences including its perception and the constitution of the self-observing subject , and thus a self-referential process, which is the embodiment of the fusion of object with subject, the seeing just mentioned [28, 13]. [3] At the level of visual perception at the neurocortex this is sustained by the self-referential topology of the Klein bottle and its paradoxical structure and is further related to Gabor wavelets and holography [48, 49, 50]. [4]

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prove below that $[\mathrm{FALSE}, \mathrm{TRUE}] = 1.\mathrm{FALSE} - 1.\mathrm{TRUE}$, (so here $n = 2$); thus the structure coefficients reduce to a vector, which defines the torsion of cognitive space, namely the vector $(-1\ \ 1)$, which we shall later associate with a normal vector to a Moebius band. For a proof of the legitimacy of this extension to cognitive space see below.

[3] Studies in mathematical psychology on visual perception have proved that there exists no such thing as a purely objective spacetime; see [13, 33, 34, 35]. Bohm has wisely thought the other way round, so to speak, to show the obvious and yet difficult to acknowledge fact that thought has an essential role in creating reality and perception [47], as the history and affairs of humankind shows reiteratively -would a better proof be left wanting-. Thus, we recover a central notion in mathematical psychology in which the construction of the geometry of visual representation depends on parameters proper to the subject [34].

[4] The Klein bottle as a 2-dimensional manifold (a Riemann surface) is not *defined* by its

Thus the conception which we shall present points to the demise of the Cartesian duality, which in logic is the Aristotelian-Boolean dual logic. Cartesian duality (also called the *Cartesian* or *epistemic cut*) appears in several guises: 1) In the formulation of the so-called mind-matter problem, separating the physical world from the *observer*, and more generally the world of *objects* from those of *subjects* (subjectivity). 2) In first-order cybernetics of observed systems, as the idealization of systems which are controlled by a detached subject vis-à-vis the integrative conception of second-order cybernetics. The latter is the cybernetics of observing systems. It is the basis of the mind-matter problem, though never addressed from the point of view of self-reference with some notable exceptions [13, 26]). 3) In the purported duality between form and function, or more generally between form and process. 4) In the duality between content and context, and an endless stream of fractures introduced by subjects, which Nature (which also evokes our nature) by no means abides to [47].

In the geometrical theory of this article, we shall show that quantum jumps are produced whenever the logarithm of the wave function -that acts as the source of the torsion singularity (a spacetime dislocation)- becomes singular on the node set of the wave function. These are the spacetime points where the wave *vanishes*. This establishes 0 as being essentially generative, and we shall see this all along the present article, in the generation of the IC and the mutual coding of light and cognitive states of the IC. We have already discussed that torsion is in distinction with the metric-based geometries of General Relativity (GR), a self-referential construction of spacetime and the subject [13]. If we remain inscribed in this realistic conception which is the daily bread of the working scientist, it is pertinent to remark that these geometrical structures with torsion fields include

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embedding in 4-dimensional spacetime as is the case of geometries for the Cartesian conception of objects, though in the usual realist approach and for computations it can be taken this embedding. In this conception, objects occupy space rather than being singularities that generate it [28]. In contrast, the Klein bottle [72] is a manifold which is self-contained, so that there is no 'exterior' or 'interior' of it, but a transformation which stems from a singularity (the hole which allows in 3D the reentrance of the Klein bottle into itself) which is the subject as already indicated, which unfolds to the whole Klein bottle to return to the singularity in a form which is a process that incorporates Bohm's explicate and implicate orders [3] as we discussed in [13]. Thus, the Klein bottle is both content and context, form and process, subject and object [28] and thus in relation to it the principle of non-contradiction is invalid [13]. This is the paradoxical being of the Klein bottle which generates matrix logic as a mathematical representation of the laws of thought and cognition. We follow Stern in calling this representation as the *Intelligence Code* (IC) [26]. This paradoxical being will produce (topo)logical superpositions states transforming into the true and false states $|1>$ and $|0>$, and viceversa, and altogether these states allow the generation of all the operators of matrix logic and thus of the IC. This produces the multivalued logic character of the IC which we shall present below [13].

the Hertz potential that yields subluminal and superluminal solutions of the Maxwell equations, and its equivalence with the Dirac-Hestenes equation in the Clifford bundle setting. They furthermore yield a theory of unification of spacetime geometries, non-relativistic and relativistic quantum mechanics [10], the weak interactions without a remaining Higgs field [43], fluid and magnetofluiddynamics [9], non-equilibrium and equilibrium statistical thermodynamics [8], the strong interactions as characterized by Hadronic Mechanics [12], and most importantly Brownian motions[7]. [5]. (Torsion is also essential to the problem of spin precession, appearing in the formulation of the classical mechanics of spinning particles submitted to gravitational fields with torsion, which does nor rely on lagrangian nor hamiltoneans [78].) So torsion is closely related to the chaotic and ordered -and generally non-equilibrium- processes which coexist in apeiron. This contrasts with Einstein's conception in which he claimed by stating that 'God does not play with dice' the elimination of chaos as it could not be framed -at that time- as geometry [7]. This seemingly dual character of apeiron has been the source for the historical record of rejection that different cultures and conceptions had with relation to this untameable Being; for a philosophical discussion we refer to the work in [28]. In [10,12] we proposed these quantum fluctuations as a source for the space anisotropy, the strong nuclear interactions and the time fields (*chronomes* [74]) discovered in tens of thousands of experiments carried out in the last fifty years [44]. In giving a first indication on the nature of torsion as related to surmounting a Cartesian conception, we point out that torsion appears with a Janus face, as the primitive distinction in the calculus of distinctions in the protologic of Spencer-Brown. In this primitive setting, by further incorporating the reentrance of a form on itself and particularly the Klein bottle, the IC is generated [13]. In this code, quantum field operators and logical operators are represented by hypernumbers, establishing thus a connection between quantum field theory, nilpotents and matrix logic. Nilpotence, which in our conception we

[5]Indeed, Brownian motions are unified into the geometrical structure of torsion geometries: the metric conjugate vector field of the trace-torsion is the drift of the Brownian motions, and the noise density is a square root of the metric which can be trivially Minkowski or Euclidean. In this setting, Brownian motions determined the torsion geometry or alternatively are determined by it. They are further related to the linear and non-linear Schroedinger equations [10] and the isotopic lift of the former in the Hadronic Mechanics theory of the strong interactions [12]. In terms of them we proposed an explanation [10] of the extraordinary experiments by Kozyrev and Nasonov in Russia -repeated by others- [69], which lead to conceive time as a physical operator related to spin, and thus ultimately to torsion fields [56], yet they were not related to Brownian motions [69]. Kozyrev's conception is currently developed in geophysics [70], chronoastrobiology [74] and in consciousness and physics studies -the Kozyrev mirrors- carried out at ISRICA-Russian Academy of Sciences [71]. These experiments manifest a pervasive action of apeiron.

more accurately call *plenumpotence* -as much as the vacuum is to be called the *plenum*-, has a crucial role in the *nilpotent universal rewrite system* [15]. In this frame for logic, these plenumpotents act as polarizations (i. e. as factorizations) of the mind apeiron defined by the logical-quantum observable given by the identically null matrix. These polarizations turn to constitute the IC, similarly to the constitution of the manifested physical world from the Brownian fluctuations -which is the essential process-structure of apeiron-, and the generative role of the zero points of the torsion generating waves. In the course of this work we shall relate the mind apeiron with the twistor representations of the photon as an extended structure which characterizes the solution of the system given by the quaternionic wave and eikonal equations.

Returning to the torsion geometry, we remark that it introduces a quantization of the apeiron, since it signifies the non-conmutativity of the infinitesimal parallel transport of two vector fields. [6] This non-commutativity manifests as the impossibility of the closure of the infinitesimal parallelogram thus constructed; this closure is achieved by introducing the torsion (the antisymmetric coefficients of the Cartan connection; see Appendix and [61, 7]) with which the parallel transport was produced in the first place. This closure has an invariant meaning since the torsion is a tensor, and thus is invariant by invertible smooth coordinate transformations. [7] This non-commutativity of parallel transport with a connection with non-null torsion, is tantamount to the quantization of the geometry of spacetim. This naturally invites to present the relation between torsion and light, and in particular the photon. So, this article will start by *this* relation to *end* with the relation between light and the laws of thought when treating the relation between twistors (which were introduced by Penrose to construct a geometrical theory of physics starting with light) and the eigenstates of the mind apeiron. Surprisingly this relation will come out from the fact that when trying to localize the photon, we shall found that the singularities of the torsion which

[6] For a detailed presentation of this we direct the reader to the Appendix in this article.

[7] This invariance is essential to the joint constitution of the world as a process and geometrical structure, and the subject. It is a mathematical instru*ment* (i.e. an instruction by the mind) by which the subject establishes an 'objectivity' of the 'outer' world while keeps its own invariance which is further projected and retrieved from this 'outer' invariance. Thus, this invariance lies at the foundation of the generation of the spacetime manifold. It further establishes form and function, content and context, outside and inside, through the fusion of the 'outer' world with that of the subject which poses-discovers this joint constitution. For a geometry thus constructed we reiterate that the metric can be the Minkowski or Euclidean metrics, this is irrelevant to the invariant process of generation of an invariant spacetime and an invariant subject. An important example of torsion geometries is given by dislocated crystals, in which dislocations are represented by the torsion tensor [5].

will be supporting it, for scalar *complex* fields does *not* provide a pointlike photon, but rather an *extended* one. On extending these fields to *quaternion* (instead of real or complex) valued functions, this is still the case. In fact we shall give a representation for the photon which will allow us to give its most general structure and still to represent it by twistors which will finally appear as eigenstates of the apeiron mind (as a non-zero polarization of it). This will lead us to conclude with the establishment of a link between light and the IC, which we here recall that is related to quantum field operators through matrix logic.

We turn to discuss light in classical physics. In his theory of gravitation that stemmed from his criticism of General Relativity (GR), V. Fock showed that light rays described by the eikonal equations of geometrical optics, were at the basis for the possibility of introducing 'objective' [8] spacetime coordinates and furthermore for the construction of a theory of gravitation based on characteristic hypersurfaces of the Einstein equations of GR. These equations being hyperbolic partial differential equations have propagating wavefronts. They arise as *singularities* of spacetime which are identical to the wavefronts singular solutions of Maxwell's covariant equations of electromagnetism: they are all characterized by the solutions of the eikonal equation. These singular propagating fields stand for the inhomogenities of the otherwise uniform spacetime that the metric based geometry of GR leads to. As already discussed, this is also a common feature with a theory of spacetime conceived in terms of Cartan geometries with torsion (which is more fundamental than curvature of the latter geometries, as the Bianchi equations show [17]) rather than the curvature produced by a metric. Without inhomogenities it is impossible to give sense to a geometrical locus as argued by Fock, and we reiterate, both are essential features generated by torsion as we argued before [12, 13]. In fact, Fock further proved that the Lorentz transformations of special relativity arise together with the Moebius (conformal) transformations as the unique solutions of the problem of establishing a relativity principle for observers described by inertial fields. It is *not* the Lorentz invariance of the Maxwell's equation what makes this invariance so important in special relativity -paving the way to a diffeomorphism invariant theory of gravitation which Einstein insisted in relating to special relativity-. For Fock, it is rather the fact that the *singular* solutions of the Maxwell equations are invariant by the Lorentz transformations and still, by the full conformal group [4]. We must recall, that already in 1910, Bateman discovered the invariance of Maxwell's equations by

[8] Fock's takes an approach based in dialectical materialism. In the philosophical approach by the present author for surmounting the Cartesian cut, the photon is not an 'objective' particle, but the very signature of the fusion of object with subject, the latter being absent in the *geometry* of GR and unacknowledged in Fock due to his mantainance of the Cartesian cut .

this fifteen dimensional Lie group. [9] The equivalence class of reference systems transformable by Lorentz transformations preserve the singular solutions propagating at a finite constant invariant speed equal to c [13]. The velocity of light waves is no longer constant for observers transformable under conformal transformations, but can be infinite [2]. Thus, for all observers related by a Lorentz transformation, if any one would identify a propagating discontinuity with velocity c, all of them would likewise identify the phenomena. Thus, while the Maxwell equations are well defined with respect to *all* diffeomorphic observers, the singular solutions with speed c are well defined for all *Lorentz* group related observers. Most importantly, the singular sets $N(\phi) = \{x \in M : \phi(x) = 0\}$ were introduced by Fock in terms of scalar fields which are solutions ϕ of the eikonal equation

$$(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 + (\frac{\partial \phi}{\partial z})^2 - (\frac{\partial \phi}{\partial t})^2 = 0, \tag{1}$$

which in the more general case of a space-time manifold provided with an arbitrary Lorentzian metric, say g, can be written as $g(d\phi, d\phi) = 0$, from which in the case of g being the Minkowski metric lead to the light-cone differential equation $(dt)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = 0$. Notice that eq. (1) is a nilpotence (or as we stated above, plenumpotence) condition on the field $d\phi$ with respect to the Lorentzian metric g. But while the Maxwell equations are invariant by these two groups (Lorentz and Moebius-conformal) transformations, one could look for propagating waves that remain solutions of the propagation equation determined by the metric-Laplace-Beltrami operator, $\triangle_g$, which we shall describe below- under arbitrary perturbations: Instead of considering solutions of the wave equation $\triangle_g \phi = 0$, which form a linear space, we want to investigate the class of solutions which are further invariant under the action of *arbitrary* (with certain additional qualifications) perturbations f (real or complex valued) acting on by composition on the ϕ's, $f(\phi)$, that verify the same propagation equation: $\triangle_g f(\phi) = 0$. Notice

[9] We see here that the Lorentz group are fundamentally related to the invariance of singularities. Consider a primitive distinction as the semiotic (i.e. through a sign) codification of torsion [13] introduced by the fusion of subject -itself a singularity albeit unacknowledged- with spacetime- . If we attach values to the 'inside' and 'outside' of this distinction, then up to a scale factor, the numerical transformation of the distinction yields the Lorentz group; see page 462 [29] and [13]. Thus through 'radar coordinates' a 2D construction of spacetime results, yet with a twist, which amounts to a built-in spinor action on the space variable. Therefore the Lorentz group is related to a valued distinction introduced by this fusion, which is a more primitive and general introduction of this symmetry group that its usual introduction in Special Relativity. Bohm noted that the establishment or recognition of distinctions is a primeval act of thought and of its further projection in the creation of a real world [47].

that in these considerations we are concerned with singularities propagating on a spacetime which is *seemingly* torsionless; this will turn out not to be the case. We start by introducing the geometrical-analytical setting with torsion.

2 RCW Geometries, Laplacians and Torsion

In this section M denotes a smooth compact orientable n-dimensional manifold (without boundary) provided with a linear connection described by a covariant derivative operator $\tilde{\nabla}$ which we assume to be compatible with a given metric g on M, i.e. $\tilde{\nabla}g = 0$. Given a coordinate chart (x^α) $(\alpha = 1, \ldots, n)$ of M, a system of functions on M (the Christoffel symbols of $\tilde{\nabla}$) are defined by $\tilde{\nabla}_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma} = \Gamma(x)^\alpha_{\beta\gamma} \frac{\partial}{\partial x^\alpha}$. The Christoffel coefficients of $\tilde{\nabla}$ can be decomposed as [7,8,9,10]

$$\Gamma^\alpha_{\beta\gamma} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} + \frac{1}{2} K^\alpha_{\beta\gamma}. \tag{2}$$

The first term in (2) stands for the metric Christoffel coefficients of the Levi-Civita connection ∇^g associated to g (which is the backbone of GR), i.e. $\left\{{\alpha \atop \beta\gamma}\right\} = \frac{1}{2}(\frac{\partial}{\partial x^\beta} g_{\nu\gamma} + \frac{\partial}{\partial x^\gamma} g_{\beta\nu} - \frac{\partial}{\partial x^\nu} g_{\beta\gamma})g^{\alpha\nu}$, and $K^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + S^\alpha_{\beta\gamma} + S^\alpha_{\gamma\beta}$, is the cotorsion tensor, with $S^\alpha_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^\kappa_{\nu\gamma}$, and $T^\alpha_{\beta\gamma} = (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta})$ the skew-symmetric torsion tensor. We are interested in (one-half) the Laplacian operator associated to $\tilde{\nabla}$, i.e. the operator acting on smooth functions, ϕ, defined on M by [10]

$$H(\tilde{\nabla})\phi := 1/2\tilde{\nabla}^2\phi = 1/2 g^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \phi. \tag{3}$$

A straightforward computation shows that $H(\tilde{\nabla})$ only depends in the trace of the torsion tensor and g, so that we shall write them as $H(g, Q)$, with

$$H(g, Q)\phi = \frac{1}{2} \triangle_g \phi + \hat{Q}(\phi) \equiv \frac{1}{2} \triangle_g + Q.\nabla\phi, \tag{4}$$

with $Q := Q_\beta dx^\beta = T^\nu_{\nu\beta} dx^\beta$ the trace-torsion one-form and where $\hat{Q}$ is the vector field associated to Q via g: $\hat{Q}(\phi) = g(Q, d\phi) = Q.\nabla\phi$, (the dot standing for the metric inner product) for any smooth function ϕ defined on M; in local coordinates, $\hat{Q}(\phi) = g^{\alpha\beta} Q_\alpha \frac{\partial\phi}{\partial x^\beta}$. Finally, $\triangle_g$ is the Laplace-Beltrami operator of g: $\triangle_g \phi = \mathrm{div}_\mathrm{g} \nabla\phi$, $\phi \in C^\infty(M)$, with div_g and ∇ the Riemannian divergence and gradient operators ($\nabla\phi = g^{\alpha\beta}\partial_\alpha\phi\partial_\beta$), respectively. Of course, on application on scalar fields, $\tilde{\nabla}, \nabla^g$ are identical: it is in taking the second derivative that the torsion term appears in the former case. Thus for any smooth function, we have

$\triangle_g\phi = (1/|det(g)|)^{\frac{1}{2}} g^{\alpha\beta} \frac{\partial}{\partial x^\beta}(|det(g)|^{\frac{1}{2}} \frac{\partial\phi}{\partial x^\alpha})$. Thus $H(g,0) = \frac{1}{2}\triangle_g$, is the Laplace-Beltrami operator, or still, $H(\nabla^g)$, the laplacian of Levi-Civita connection ∇^g given by the first term in eq. (2). The connections $\tilde{\nabla}$ defined by a metric g and a purely trace-torsion Q are called RCW (after Riemann-Cartan-Weyl) connections with Cartan-Weyl trace-torsion one-form, hereafter denoted by Q [7-10].

3 Quantum Jumps and Torsion

The following section follows our work [11]. In the following we shall take g to be a Lorentzian metric on a smooth time-oriented space-time four-dimensional manifold M which we assume compact and boundaryless; we have the associated volume n-form given by $\mathrm{vol}_g = |\det(g)|^{\frac{1}{2}} dx^1 \wedge \wedge dx^2 \wedge dx^3 \wedge dx^4$, where (x^1, x^2, x^3, x^4) is a local coordinate system. This is a more general case with regards to what the essential condition for the aether is, the Minkowski metric, which is thus incorporated into this setting, though we shall later consider compact submanifolds of Minkowski space, namely the Klein bottle. [10]

The solutions of the wave equation constitute a linear space. Furthermore, the germ of solutions of the wave equation in a neighborhood of a point form a linear space. Thus , the algebra generated by a single solution of the wave equation

$$\triangle_g\phi = 0, \tag{5}$$

consists of solutions of this equation if and only if ϕ satisfies in addition the eikonal equation of geometrical optics

$$(\nabla\phi)^2 := g(\nabla\phi, \nabla\phi) = 0. \tag{6}$$

Indeed, if f is twice continuously differentiable (we shall say a C^2 function) and ϕ is real-valued, or still, if f is analytic and ϕ is complex valued, then the following identity is valid

$$\triangle_g(f(\phi)) = f'\triangle_g\phi + f''(\nabla\phi)^2. \tag{7}$$

[10]In spite of the greater generality, it is this case what we have in mind: The Klein bottle as a self-contained manifold (in fact a Riemann surface) produced by a distinction in an homogeneous plane, which thus establishes a *limited* closed domain lifting to 3d, reenters itself by producing an essential singularity in its structure [72]; see footnote no. 3 above. So instead of a Cartesian aether (a conceptual impossibility [28, 13]), we have a bounded self-referential one. The boundedness is *quantized* by a Planck constant, which we recall that it may have different values, including cosmological scales [30, 31].

The solutions of the system of equations

$$\begin{aligned} \triangle_g \phi &= 0 & (8)\\ (\nabla\phi)^2 &= 0, & (9) \end{aligned}$$

are called *monochromatic waves*. They represent pure light waves; we already discussed their relevance. A set of monochromatic waves having the structure of an algebra, will be called a *monochromatic algebra*. In Fock's approach, they are called *electromagnetic signals* [4]. Notice that the eikonal equation is a nilpotence condition for $d\phi$, the differential of ϕ, or equivalently its gradient, $\nabla\phi$, under the square multiplication defined by the metric. From the identity

$$e^{-i\phi}\triangle_g e^{i\phi} = i\triangle_g\phi - (\nabla\phi)^2, \tag{10}$$

we obtain , if $\triangle_g\phi = 0$,

$$(\nabla\phi)^2 = -e^{-i\phi}\triangle_g e^{i\phi}, \tag{11}$$

Let us consider the mapping $\phi \to e^{i\phi} = \psi$ which transforms the linear space of solutions of the wave equation into a multiplicative $U(1)$-group, in which the kinetic energy integrand in the lagrangian functional $(\nabla\phi)^2$ is transformed into $-\frac{\triangle_g\psi}{\psi}$, which has the familiar form of the quantum potential of Bohm, yet in a relativistic domain [3, 7]. If the ϕ are real valued, then the ψ are bounded and we can embed the above group in the Banach algebra under the supremum norm that it generates under pointwise operations and further completion [1]. To distinguish between them we call the original linear space the functional phase space $\mathbf{S}$ and the Banach algebra defined above as the algebra of wave states $\mathbf{A}$, or simply the functional algebra of states. It is simple to see that the critical points of the functional

$$J(\psi) = \int \frac{\triangle_g\psi}{\psi}\mathrm{vol}_g, \tag{12}$$

are those ψ which satisfy

$$\triangle_g \ln\psi = 0, \tag{13}$$

i.e., those whose phase function satisfy the wave equation. Those intrinsic states will be called *elementary states*. The new representation has two advantages over the original one. It is richer in structure and in elements, as $\mathbf{S}$ is mapped into a subset of the set of invertible elements Ω of $\mathbf{A}$. Thus, by taking the logarithm pointwise on the elements of Ω, we obtain an enlargement of $\mathbf{S}$ by possibly *multivalued* functions. The second advantage , that actually justifies the whole construction, is that the integrand of the lagrangian $-\frac{\triangle_g \psi}{\psi}$, when integrated, exhibits jumps across the boundary $\partial\Omega$ of Ω. These jumps correspond to kinetic energy changes. In the interpretation of the integrand as a quantum potential, they represent a change due to the *holographic* information of the system present in the whole Universe; see [3, 39]). Let $\mathbf{A}$ be a Banach algebra of continuous complex-valued functions defined on a four-dimensional Lorentzian manifold (M, g), containing the constant functions, closed under complex-conjugation, with the algebraic operations defined pointwise and the supremum norm and containing a dense subset $\mathbf{A_2}$ of C^2 functions which are mapped by the Laplace-Beltrami operator $\triangle_g$ into $\mathbf{A}$. Assume further $f \in \mathbf{A}$ is invertible with inverse $f^{-1} \in \mathbf{A}$ if and only if $\inf_{\mathrm{M}}|f(x)| > 0$. The set of invertible elements is denoted by Ω. Furthermore, assume a positive linear functional, denoted by λ such that $\lambda : \mathbf{A_2} \cap \Omega \to \mathbf{C}$ (the complex numbers) defined by

$$\lambda(\phi) = \int \frac{\triangle_g \phi}{\phi} \mathrm{vol_g} \tag{14}$$

The critical elements of λ are those u such that

$$\mathrm{div}(\frac{\mathrm{grad}u}{u}) = 0, \quad i.e. \; \frac{\triangle_g u}{u} - (\frac{\mathrm{grad}u}{u})^2 = 0. \tag{15}$$

If the linear functional is strictly positive, i.e. $\lambda(\phi) = 0$ if and only if $\phi \equiv 0$, these two identities are to hold in $\mathbf{A}$, otherwise in the sense of the inner product defined by λ on $\mathbf{A}$. By eq. (15) the set C of critical points of λ is clearly a subgroup of Ω. The monochromatic functions of $\mathbf{A}$ are as before, those $w \in \mathbf{A}_2$ satisfying the system of eqs. $(8, 9)$ and their set is denoted by $\mathbf{M}$. From eq. (13) the composition function given by $f(w)$ belongs to $\mathbf{M}$ again if f is an analytic function on a neighborhood of the set of values taken by w on M. Since by eq. (15) $\mathbf{M} \cap \Omega \subset C$, we have that $uf(w) \in C$ if $w \in \mathbf{M}$ and $f(w) \in \Omega$. The spectrum $\sigma(v)$ for any $v \in \mathbf{A}$, is defined by $\sigma(v) = \{z \in C / |v - ze| \notin \Omega\}$ and therefore, by a previously assumed property, is the closure of the set of values $v(x)$ taken by v on M [1]. It is obviously a compact non-void subset of C. Ω has either one or else infinitely many maximal connected components, of which Ω_0 is the one containing the identity, e, defined by $e(x) \equiv 1$. Two elements f, h belong to the same component of Ω, if and only if $fh^{-1} \in \Omega_0$. Further, $f \in \Omega - \Omega_0$ if and only if its spectrum $\sigma(f)$ separates 0 and ∞. The logarithm function, as a mapping from $\mathbf{A}$ into $\mathbf{A}$ is defined only on Ω_0 [1]. With these preliminaries, we can now show that the quantum jumps arise as a generalized form of the standard argument principle of complex analysis. [11]

Theorem Let $u \in C, w \in \mathbf{M} \cap \Omega$, i.e, it is an invertible monochromatic function. Denote by $H_1, H_2, \ldots,$ the maximal connected components of the complement of $\sigma(w)$. Then there exists fixed numbers $q_i, i = 1, \ldots,$ depending on u and w only, such that for any function $f(z)$ analytic in a neighborhood of $\sigma(w)$ and with no zeros in $\sigma(w)$, we have

$$\lambda(uf(w)) = \lambda(u) + \sum_i (N_i - P_i)q_i, \tag{16}$$

where N_i, P_i are the number of zeros and poles, respectively, of f in H_i, $i = 1, 2, \ldots$. In particular choosing $\alpha_i \in H_i$, the q_i are given by

$$q_i = 2\int g(\frac{\nabla u}{u}, \frac{\nabla w}{w - \alpha_i})\mathrm{vol_g}, i = 1, 2, \ldots. \tag{17}$$

Proof.[12] Let $f = f(w) \in \mathbf{M}$ with $f(z)$ as in the hypothesis. A computation yields

$$\frac{\triangle_g(uf)}{uf} - \frac{\triangle_g u}{u} = 2g(\frac{\nabla u}{u}, \frac{\nabla f}{f}), \tag{18}$$

which we note that it is another way of writing

$$\frac{\triangle_g(uf)}{uf} = \frac{1}{u}H(g, \frac{df}{f})(u), \tag{19}$$

where we have introduced in the r.h.s. of eq. (19) the laplacian defined in eq. (4) by a RCW connection defined by the metric g and the trace-torsion $\mathrm{Q} = \frac{df}{f}$.

[11] The following result is a simpler geometrical version of a theorem proved by Nowosad in the more intrincate setting of non-compact manifolds and functionals on generalized curves in L.C. Young's calculus of variations for curves with velocities having a probability distribution (Young measures)[6]. In our approach surmounting the Cartesian cut, we were interested in a particular Riemann surface, the Klein bottle. We may consider the embedding of this surface in a compact submanifold of Minkowski space and we are in the situation of the theorem below without the need of intrincate variational problems nor the full Minkowski space. The latter in its unboundedness corresponds to a conception of spacetime which is associated to the Cartesian approach [28] and its epistemic cut that is surmounted by considering torsion as a self-referential construction of spacetime, logic and cognition [13].

[12] An example. Take a compact submanifold of Minkowski space and plane waves with adequate boundary periodicity conditions. Take $u = e^{ik.x}, w = e^{ik_0.x}, k_0^2 = 0, k_0.k \neq 0$ and the spectrum $\sigma(w) = S^1$, where S^1 is the unit circle; then $\lambda(u) = -k^2$ (minus the mass squared) and eq. (16) becomes $-\lambda(e^{ik.x}f(e^{ik_0.x})) = k^2 + 2(k_0.k)(N - P)$, where N and P are the number of zeros and poles of f inside of the unit circle.

Integrating eq. (18) yields,

$$\lambda(uf) - \lambda(u) = 2\int g(\frac{\nabla u}{u}, \frac{\nabla f}{f})\mathrm{vol_g}. \tag{20}$$

In particular this shows that q_i in eq. (17) are well defined. From (20) one gets directly

$$\begin{aligned}\lambda(ufh) - \lambda(u) &= [\lambda(uf) - \lambda(u)] + [\lambda(uh) - \lambda(u)] &(21)\\ \lambda(uf^{-1}) - \lambda(u) &= -[\lambda(uf) - \lambda(u)], &(22)\end{aligned}$$

where $h = h(w)$ as well as we recall $f = f(w)$, are the composition functions, from now onwards. Now if $f \in \Omega_0$, then $\mathrm{ln} f \in \mathbf{A}$ and $\nabla \mathrm{ln} f = \frac{\nabla f}{f}$, which substituted in (20) gives, upon integration,

$$\lambda(uf) - \lambda(u) \quad = \quad 2\int g(\frac{\nabla u}{u}, \nabla \mathrm{ln} f)\mathrm{vol_g} = -2\int f \mathrm{div_g}(\frac{\nabla u}{u})\mathrm{vol_g} = 0, \tag{23}$$

by eq. (15). Hence

$$\lambda(uf) = \lambda(u), \quad \text{if} \quad f \in \Omega_0. \tag{24}$$

If now f, h belong to the same component of Ω we can write $uh = (uf)(hf^{-1})$, and since $hf^{-1} \in \Omega_0$, the previous result yields

$$\lambda(uh) = \lambda(uf). \tag{25}$$

This shows that $\lambda(uf(w))$ is locally constant in Ω as f varies in the set of analytic functions. Let now $f(z) = z - \nu$ with $\nu \in H_i$. Then $z - \nu$ can be changed analytically into $z - \alpha_i$ without ν leaving H_i, which means that $w - \nu e$ and $w - \alpha_i e$ are the same connected component of Ω, with $e \equiv 1$. Therefore from eqs. $(25, 20)$ and eq. (17) follows that

$$\lambda(u(w - \nu e)) - \lambda(u) = q_i, \tag{26}$$

and by eq. (22)

$$\lambda(u(w - \nu e)^{-1})) - \lambda(u) = -q_i. \tag{27}$$

On the other hand, if ν belongs in the unbounded component of the complement of $\sigma(w)$, we may let $\nu \to \infty$ without crossing $\sigma(w)$ so that

$$\begin{aligned}\lambda(u(w - \nu e)) - \lambda(u) &= 2\int g(\frac{\nabla u}{u}, \frac{\nabla w}{w - \nu e})\mathrm{vol_g}\\ &= \mathrm{lim}_{\nu\to\infty} 2\int g(\frac{\nabla u}{u}, \frac{\nabla w}{w - \nu e})\mathrm{vol_g} = 0. &(28)\end{aligned}$$

Therefore, if $f(z) = c_0\Pi_{i=1}^{N}(z-a_i).\Pi_{j=1}^{p}\frac{1}{z-b_j}, c_0 \neq 0, a_i, b_j \notin \sigma(w)$, then eq. (16) follows from eqs. $(21, 27, 28)$ In the general case , if $f(z)$ is an holomorphic function in a neighbourhood of $\sigma(w)$,without zeros there, we can find a rational function $r(z)$ such that

$$|f(z) - r(z)| < \min_{\sigma(w)}|f(z)| \text{in} \sigma(w), \tag{29}$$

by Runge's theorem in complex analysis. Then, $r(z)$ has no zeros in $\sigma(w)$ too, and $r(w)$ and $f(w)$ are in the same component of Ω, so that eq. (16) holds for $f(w)$ too. The proof is complete.

Observations. The quantization formula (16) tells us how the basic functional changes when we perturb the elementary state u into $uf(w)$ with f analytic near and on $\sigma(w)$. Changes occur only when zeros or poles of $f(z)$ reach and eventually cross the boundary of $\sigma(w)$, and these changes are integer multiples of fixed quanta q_i, each one attached to the hole H_i whose boundary is reached and crossed, while u, v remain fixed. Two more aspects are important. The first one being that the actual jump is measured modulo the product of the q_i by a classical difference (where by classical we stress we mean that it is the substraction, in distinction of the quantum difference given by the commutator of operators) of poles and zeros. At the level of second quantization quantum jumps appear in terms of the difference of the creation and annihilation operators. These in turn define the TIME operator in matrix logic in which the commutator of the FALSE and TRUE logical operators coincide with their classical difference, establishing thus a non-null torsion in cognitive space . The second aspect is the actual form of the q_i which are given by integrating the internal product of the trace-torsion one-form $Q = \frac{du}{u}$ defined by the critical state u, with another almost logarithmic differential of the form $dw/(w - \alpha_i)$. Finally, let C_u denote the linear operator $h \to uh, h \in \mathbf{A}, u \in \Omega$. The very simple analysis above hinges on the fact that $C_u^{-1} \circ \triangle_g \circ C_u - C_{\frac{\triangle_g u}{u}}$ is a derivation on the germ $\mathbf{F}(w)$ of functions of w (see eq. (14) and still eq. (19) to see how it is related to the torsion geometry), which are analytic in a neighbourhood of $\sigma(w)$, and it could have been performed abstractly without further mention to the special case under consideration. The general abstract theory of variational calculus extending the functional λ for quantum jumps when specialized to second order differential operators, say $\triangle_g$ or still $H(g, Q)$, shows that the condition $w \in \mathbf{M}$ in not only sufficient but also necessary in order to the quantum behaviour of λ occur [6]. Let us see next the relation with RCW geometries.

The set of linear mappings $C_{f^{-1}}$ of $\mathbf{A}$ defined by $h \to f^{-1}h, h \in \Omega$, f defined on M, is a group which maps each connected component of Ω onto another one. In terms of functions defined on M it changes locally the scale of the functions, i.e. the ratio of any function at two distinct points is changed in a given proportion, and it therefore a gauge transformation of the first kind. Under this transformation we have that

$$\triangle_g \rightarrow C_{f^{-1}}\triangle_g C_f = \triangle_g + 2\frac{\nabla f}{f}.\nabla + \frac{\triangle_g f}{f} = 2H(g, \frac{df}{f}) + 2V_f, \tag{30}$$

where $H(g, \frac{df}{f})$ is the RCW laplacian operator of eq. (4) with trace-torsion 1-form $Q = \frac{df}{f}$ and $V_f = \frac{\triangle_g f}{f}$ is the relativistic quantum potential defined by f^2 [10]. Now noting that for vectorfields $A = A^i\partial_i, B = B^i\partial_i$, with $A^i, B^i, i = 1, \ldots, 4$ complex valued functions on M, with the hermitean pairing defined by the metric g on M, i.e. $\int g(\bar{A}, B)\mathrm{vol_g} = \int g(B, \bar{A})\mathrm{vol_g}$ so that $A^\dagger = \bar{A} = \bar{A}^i\partial_i$. Therefore for the gauge transformation $d \rightarrow d + \frac{df}{f}$, since $\triangle_g = -d^\dagger d$ (see [9,10,15]), we further have the transformation

$$-d^\dagger d \rightarrow -(d + \frac{df}{f})^\dagger(d + \frac{df}{f}) = -(d^\dagger + \overline{(\frac{df}{f})}.)(d + \frac{df}{f}). \tag{31}$$

where $d^\dagger$ is the adjoint operator,the codifferential, of d with respect to this hermitean product so that $d^\dagger = -\mathrm{div_g}$ on vectorfields [15]. If we assume that $\overline{(\frac{df}{f})} = -\frac{df}{f}$, so that $|f(x)| \equiv 1$ and thus f is a phase factor, $f(x) = e^{i\phi(x)}$, i.e. a section of the $U(1)$-bundle over M then the r.h.s. of eq. (31) can be written as

$$\begin{aligned} -(d^\dagger - \frac{\nabla f}{f}.)(d + \frac{df}{f}) &= \triangle_g + 2\frac{\nabla f}{f}. + (\frac{df}{f})^2 + \frac{\triangle_g f}{f} - (\frac{df}{f})^2 \\ = (\triangle_g + 2\frac{\nabla f}{f}.) + \frac{\triangle_g f}{f} &= 2H(g, \frac{df}{f}) + \frac{\triangle_g f}{f} = C_{f^{-1}} \circ \triangle_g \circ C_f. \end{aligned} \tag{32}$$

Consequently, if f is a phase factor on M, then under the gauge transformation of the first kind $h \rightarrow f^{-1}h$, the change of $\triangle_g$ into $C_{f^{-1}} \circ \triangle_g \circ C_f$ can be completely determined by the transformation $d \rightarrow d + \frac{df}{f}$ which is nothing else than the gauge-transformation of second type, from the topological (metric and connection independent) operator d to the covariant derivative operator $d + \frac{df}{f}$, of a RCW connection whose trace-torsion is $\frac{df}{f}$, equivalent to the gauge transformation $d \rightarrow d + A$ in electromagnetism [15].

In summary, when f is a phase factor, the gauge transformations of the first and second type are equivalent, and gives rise to the exact Cartan-Weyl 1-form. If we further impose on f the condition similar to the one placed for the electromagnetic potential 1-form, A, to satisfy the Lorenz gauge $\delta A = 0$, i.e. $\delta(\frac{\mathrm{grad} f}{f}) = 0$, we find that this is nothing else than the condition on f to be an elementary state i.e. a critical point of the the functional $\lambda(f)$ given by (14). Therefore, when f is a phase factor, both the first and second kind of gauge transformations are equivalent and they give rise to a Cartan-Weyl one-form $Q = \frac{df}{f}$. When $\frac{df}{f}$ cannot be written globally as $d\ln f$, f is said to be a non-integrable phase factor. When f belongs to the algebra $\mathbf{A}$, this is equivalent to saying that f does not have a logarithm in $\mathbf{A}$, which means that $f \in \Omega - \Omega_0$. In any case, the 2-form of intensity $F = d(\frac{df}{f})$ is always identically 0 because $\frac{df}{f}$ can be *locally* written as $d\mathrm{Log} f$, where Log is a pointwise locally defined logarithm determination.[13] Consider now all the connected components Ω_α of Ω. Any such component can be transformed into Ω_0 by a gauge-transformation of the first kind: it suffices to take $f \in \Omega_\alpha$ and consider $h \to f^{-1}h$, which is indeed a diffeomorphism of Ω. This choice of the component, is a choice of gauge, and of course, there is no preferred gauge. That is, the topological operator d of one observer becomes the covariant derivative operator $d + \frac{df}{f}$ of a RCW connection for the other observer. We can interpret the difference of gauges as being equivalent to the presence of the trace-torsion 1-form $\frac{df}{f}$ in the second's observer referential. However as the electromagnetic 2-form $F \equiv 0$, this is an instance of the Bohm-Aharonov phenomena: non-null effects associated with identically zero electromagnetic fields. That there are non-null effects is checked by our previous analysis of the functional $\lambda(uf(w))$, where u is any elementary state and f, besides being a phase factor, is also monochromatic. In this case λ, which is locally constant depends on which $\Omega_\alpha f$ belongs to, that is to say, on the choice of the gauge. Finally, according to the two ways of interpreting a linear operator (as a mapping on the vector space or as a change of referential frames) we have two possibilities. Indeed let $w \in \mathbf{M}$ and let $f_t(w), t \in [0,1]$ with $f_t(z)$ analytic in a neighbourhood of $\sigma(w)$, be a continuous curve on $\mathbf{A}$. For any $u \in C$ we consider the curve of elementary states $uf_t(w)$; we described in eq. (16) the behaviour of $\lambda(uf_t(w))$ along this curve. In particular we considered $uf_t(w)$ as a perturbation, or excitation, of u evolving in time (here time may not be the time coordinate of a Lorentzian manifold but the universal evolution parameter introduced first in quantum field theory with other important current formulations; see [16].) We can also regard $u \to C_{f_t(w)}u$ as a continuous curve of gauge transformations of first kind acting on a fixed elementary state u, which, when f_t crosses $\partial\Omega$, determines a change of gauge. When that happen f_t cannot be made a phase factor for all t obviously, so that no electromagnetic interpretation can be given all along the evolution in t. However

[13] The relation between Cartan torsion, singularities and dislocations in condensed matter physics is well known [5].

if, say, the initial states f_0 and f_1 are phase factors (i.e. $|f_i(x)| \equiv 1, i = 0, 1$), this change of gauge is equivalence to the appearance of a non-trivial trace-torsion one-form, which we can interpret as an electromagnetic potential, between the initial and final states. In any of these interpretations a non-null effect is detected by a jump in λ as given by eq. (16). This quantum transition is interpreted in the first case as an excitation of the state u. In the second case as a change of gauge of u. This materializes by the appearance of the corresponding Cartan-Weyl one-form as an electromagnetic Arahonov-Bohm potential with zero intensity and non-null effects. Thus, in this interpretation, *quantum jumps are the signature of a non-trivial geometrical structure, the appearance of torsion.*

Finally we examine the dimensions of singular sets $N(f)$ of monochromatic functions. Recall that a C^2 real or complex-valued function f defined on (M, g) is a monochromatic wave, $f \in \mathbf{M}$, if it satisfies the system given by eqs. $(8, 9)$. In the real-valued case, all C^2 functions of f, and in the complex case, all analytic or anti-analytic functions of f belong to $\mathbf{M}$ again, by eq. (7) (we changed here our notation there, pointing precisely to $f = f(w)$ for $w \in \mathbf{M}$, as above) . If f is real, smooth and $df \neq 0$, then $N(f)$ is locally three-dimensional. If it is complex and $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are functionally independent $N(f)$ is two-dimensional. Yet the Newtonian picture of a photon as an isolated point-like singularity moving with the speed of light in the vacuum, requires a one-dimensional singular set $N(f)$. Can we achieve this by going to hypercomplex, say quaternionic functions, or still Musès' hypernumbers which are rich in divisors of 0? The answer to the former question is negative; in the quaternionic framework, the photon is a propagating three-dimensional singularity with lower dimensional singularities, but still undivisely extended. We shall present this in the next section. For closing remarks we note that quantum jumps were obtained here in terms of the quantum potential which stands for an holographic in-formation of the whole Universe. In considering the semiclassical theory of gravitation, quantum jumps produce discontinuities in the energy-momentum tensor. These jumps produce a cosmological time associated with a quantum-jumps, in a *global* canonical decomposition of spacetime; see arXiv:gr-qc/0303046v1. [14]

[14] This work departs from the incompleteness of the Cauchy problem for the Einstein equations of GR: They provide only six equations for the ten components of the metric. For curved spacetime, it is proved that diffeomorphism invariance of the solutions of the Einstein equation is not valid; only in the case of Ricci flat spacetime this is assured. This underdetermination is resolved by the canonical complementary conditions. In the semiclassical approach they are provided by nonlocal quantum jumps; instead, in Fock's theory they are provided by four equations as eq. (5), the so-called harmonic coordinates. Thus, quantum jump nonlocality is essential for GR, it occurs in nonempty spacetime where the underdetermination problem arises

4 Monochromatic Hypercomplex Functions

This section will deal with the problem of the non-pointlike extended structure of the photon by expanding the field of **C**-valued to quaternion-valued propagating waves verifying the plenumpotence eikonal equation. Two results will appear: Firstly that the node set of these waves reduces to a *single* set, and furthermore, the *generic* form of these waves (Theorems 1 & 2 below), which will later play a crucial role in finding its spinor and twistor representations, which in this article will finally be associated with the Intelligence Code. Our presentation will be highly technical, following [6] and can be skipped - in a first reading if wished- to focus in the statements of these theorems.

A system S of hypercomplex numbers is a finite-dimensional vector space over **R** (or **C**) on which multiplication of any ordered pairs of elements is defined , taking into S it again, and being distributive with respect to vector addition. If $\{e_1, \ldots, e_n\}$ for a basis of this vector space we get

$$e_i e_j = \sum_{k=1}^{n} c_{ijk} e_k, (i, j = 1, \ldots, n), \tag{33}$$

with $c_{ijk} \in \mathbf{R}(or\mathbf{C})$. The constants c_{ijk} are called the constants of the multiplication table of S with respect to a given base, where we still have denoted the product by the juxtaposition. These constants are arbitrary and once fixed, define the multiplication according to the above rule. The product in S is associative if and only if $e_i(e_j e_k) = (e_i e_j)e_k$, for all $i, j, k = 1, \ldots, n$ and this imposes conditions on the c_{ijk}. Furthermore, S has a principal unit u, i.e. an element such that $ux = xu = x, , \forall x$, if and only if there are numbers $\alpha_1, \ldots, \alpha_n$ such that $\sum_i \alpha_i c_{ijk} = \delta_{jk(j,k=1,\ldots,n)}$. In general, S need not be commutative. However, the algebra generated by a single S-valued function f defined on M is always commutative provided S is associative. We shall assume next,that S is associative and has a principal unit. In this case S is isomorphic to a subalgebra of the algebra M_n of n times n matrices over **R** (or **C**) through the correspondence

$$a = a_1 e_1 + \ldots + a_n e_n \to C_a \in M_n \tag{34}$$

which associates with an element $a \in S$ the matrix C_a of the linear operation $x \to ax$ in S, with respect to the basis $\{e_1, \ldots, e_n\}$. Thus, C_a is given explicitly

and actually solves this problem. Furthermore, quantum jumps lead to a Universe with complete retrodiction in which only partial prediction is possible; see arXiv:gr-qc/0303046v1.

by

$$(C_a)_{jk} = \sum_{i=1}^{n} a_i c_{ijk} (j, k = 1, \ldots, n). \tag{35}$$

S may contain zero-divisors, i.e. non-invertible elements other than 0. An element a is non-invertible if and only if $\det C_a = 0$, which means that at least one of the eigenvalues λ_i of C_a is 0. Therefore, if f is an S-valued function on M, then $N(f)$ defined as the set of points $x \in M$ where $f(x)$ is not invertible, is the set of points where at least one of the (possibly complex) eigenvalues λ_i of C_a is zero. If the λ_is are locally smooth functions, $N(f)$ will be the finite union of the sets $N(\lambda_i)$, each of which will be at least two-dimensional (over the real numbers). Hence, so will $N(f)$ be at least two-dimensional (over the reals). Thus, we have proved that it is *impossible* to localize a photon to be a *one*-dimensional Newtonian singularity, by the provision of taking a hypercomplex field. Our task goes further to give a structure form of monochromatic quaternionic functions.

We shall say that an S-valued function f is locally smooth if its components f_i *and* eigenvalues λ_i can be chosen locally as smooth functions on M. Clearly the condition

$$\triangle_g f = e_1 \triangle_g f_1 + \ldots + e_n \triangle_g f_n = 0, \tag{36}$$

implies that

$$\triangle_g f_i = 0, \forall i = 1, \ldots, n. \tag{37}$$

As the entries of C_f are given by $\sum_{i=1}^{n} f_i c_{ijk}, j, k = 1, \ldots, n$ and the c's are constant, this implies that all the entries of C_f satisfies this equation again. The hypothesis that $f \in \mathbf{M}$ means that any entire function ϕ of f with real (or complex) coefficients, satisfies

$$\triangle_g \phi(f) = 0. \tag{38}$$

Combining this with the previous remark and with the fact that

$$\mathrm{tr} C_{\phi(f)} = \sum_i \phi(\lambda_i), \tag{39}$$

we get

$$\triangle_g \sum_i \phi(\lambda_i) = \sum_i [(\phi'(\lambda_i) \triangle_g \lambda_i + \phi''(\lambda_i)(\nabla \phi)^2)] = 0. \tag{40}$$

at the point p. Let n_p be the number of distinct eigenvalues of $C_{\phi(f)}$ at the point $p \in M$. By taking $\phi(\lambda) = \frac{\lambda^p}{p}, p = 1, \ldots, 2n_p$, in turn in eq. (40) we obtain $2n_p$ linear homogeneous equations at a point p in M. The unknowns are the sums $\sum_i \triangle_g \lambda_i^{(k)}, \sum_i (\nabla \lambda_i^{(k)})^2, k = 1, \ldots, n$ where $\lambda_i^{(k)}$ are the original eigenvalues grouped by the condition $\lambda_i^{(k)} = \lambda_k$ at $p, i = 1, \ldots, k = 1, \ldots, n_p$ (so called $\lambda(p)$-groups). Since the above system has non-zero determinant we get the $2n_p$ conditions holding at p,

$$\sum_i \triangle_g \lambda_i^{(k)} = 0, \sum_i (\nabla \lambda_i^{(k)})^2 = 0, k = 1, \ldots, n_p. \tag{41}$$

Simple eigenvalues therefore satisfy $\triangle_g \lambda = 0$, $(\nabla \lambda)^2 = 0$, i.e. $\lambda \in \mathbf{M}$. So do obviously the multiple eigenvalues of a group of eigenvalues that are coincident in an open set and remain distinct from the other in that set. More general situations arise as limiting combinations of both these cases. We therefore conclude that, generically speaking, the eigenvalues of an S-valued monochromatic function should be monochromatic itself. This therefore implies that

$$N(f) = \bigcup_{i=1}^{n} N(\lambda_i), \quad \text{with} \lambda_i \in M, i = 1, \ldots, n, \tag{42}$$

Consequently, the analysis of singular sets of monochromatic S-valued functions reduce to the analysis of those singular sets of real (complex)-valued functions defined on M.

We will now show that $N(f)$ reduces to a single set, $N(\lambda)$, λ real or complex, for all possible S-valued functions if and only if S is a division algebra over $\mathbf{R}$ or $\mathbf{C}$, i.e. S is either $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, where $\mathbf{H}$ denotes the real quaternions (Hurwitz theorem) [18]. To show this we need the following facts. Any linear associative algebra has a uniquely determined maximal nilpotent ideal (its radical, R) and is isomorphic to the sum of R with the semisimple algebra S/R. Each semisimple algebra is the direct sum of simple algebras, and Cartan's fundamental theorem says that the simple algebras over $\mathbf{R}$ are just the matrix algebras $M_m(\mathbf{R})$, $M_m(\mathbf{C})$ and $M_m(\mathbf{H})$, and over $\mathbf{C}$ just $M_m(\mathbf{C})$, up to isomorphisms. In particular from this follows that the only real division algebras, i.e. real algebras with no zero divisors are $\mathbf{R}, \mathbf{C}$ and $\mathbf{H}$, and the only complex one is $\mathbf{C}$ itself [19].

To prove the above claim we make the following observations:

1. If $a \in S$ is invertible then so is $a + r$, for any $r \in R$, and viceversa, because $(1 - r')^{-1}$ exists if $r' \in R$ and is given by $1 + r' + \ldots + r'^n$ and so therefore so does

$$(a + r)^{-1} = (1 + a^{-1} r)^{-1} a^{-1}. \tag{43}$$

2. If $p(x)$ is a polynomial in the indeterminate x then,

$$p(a+r) = p(a) + r', r' \in R, \tag{44}$$

and so also for any analytic function f.

3. Take any basis of S formed by a basis $\{e_1, \ldots, e_p\}$ of R and a basis $\{e_{p+1}, \ldots, e_n\}$ of a linear space K complementary to R in S. Since R is an ideal, we have $e_i e_j \in R$ if not both i, j are bigger than p. This means in particular, if $i, j, k > p$ then $e_i e_j e_k$ has the same last $n-p$ coefficients that it would have if we had disregarded in the product $e_i e_j$ its coefficients with respect to $e_1, \ldots, e_p$ in the given basis (by induction, this extends to any number of factors). Therefore , if we define a new product in K given by the original multiplication table restricted to indices $i, j > p$ leaving the vector addition unmodified, K is then a concrete representation of S/R. Furthermore if p is a polynomial, $a \in K$ and $r \in R$ then

$$\pi p(a+r) = \tilde{p}(a), \tag{45}$$

where π is the projection on K along R and $\tilde{p}$ is the same polynomial p but computed on the element $a \in K$ with the restricted multiplication table defined above.

Therefore, let q be a smooth monochromatic S-valued function on (M, g). Decomposing it according to the subspaces K and R we get $q = a + r$, with smooth functions $a \in K, r \in R$. We claim that a is a monochromatic K-valued function under the restricted multiplication table. Indeed, by hypothesis $\triangle_g p(a+r) = 0$ for any polynomial p and this holds if and only if each of the coefficients of $p(a+r)$ with respect to the basis $e_1, \ldots, e_n$ satisfies the same equation. But this implies, in particular, $\triangle_g \pi p(a+r) = 0$ and by the last equation, then $\triangle_g \tilde{p}(a) = 0$, which proves the claim.

Combining remarks $1, 2, 3$

$$N(p(a+r)) = N(p(a)) = N(\pi p(a)) = N(\tilde{p}(a)). \tag{46}$$

for polynomials and so for analytic functions as well. This means that passing from S into K with the restricted multiplication , preserves the monochromatic functions and their singular sets. Since K with the new multiplication is semisimple, the claim above now follows from the fact that it is then a direct sum of the matrix algebras given by the Cartan theorem. In the case of division algebras, as $\mathbf{R}, \mathbf{C} \subset \mathbf{H}$, it suffices that we obtain the general form for the quaternion-valued monochromatic function, because the real and complex ones are then obtained by restriction and complexification. Further the knowledge of $N(\lambda)$ for real and complex monochromatic λ gives $N(f)$ for general hypercomplex functions f, according to the decomposition formula (42) above.

4.1 Monochromatic Quaternion-Valued Functions

Let us introduce the quaternionic units $\vec{i_1}, \vec{i_2}, \vec{i_3}$ given by the multiplication rules

$$\begin{aligned} \vec{i_1}\vec{i_2} &= \vec{i_3}, \vec{i_2}\vec{i_3} = \vec{i_1}, \vec{i_3}\vec{i_1} = \vec{i_2} \\ \vec{i_j}\vec{i_k} &= -\vec{i_k}\vec{i_j}, k \neq j, \vec{i_k}^2 = -1, j,k = 1,2,3. \end{aligned} \tag{47}$$

Notice here that we could chose here the logical quaternions introduced in matrix logic [13], and thus the structures we shall produce below, can be conceived as spacetime structures which are both 'inner' and ' outer' representations of the self-referential character of photons (though the neutrino is also considered below). We shall introduce the notation $(\phi, \psi) \in \mathbf{M}$ to mean that $\phi, \psi \in \mathbf{M}$ (i.e. they satisfy eqs. $(7,8)$) *and* furthermore

$$g(\nabla\phi, \nabla\psi) = 0, \tag{48}$$

which is the requirement that any algebraic combination of ϕ, ψ belong in $\mathbf{M}$ as well. It will also be assumed that ϕ and ψ are functionally independent, to rule out the trivial cases. We then have the following theorem.

Theorem 1. Any monochromatic quaternion valued function F defined on (M, g) is determined by a triple of real valued functions (ϕ, f, ρ) such that

$$(\phi, f + i\rho) \in \mathbf{M}, \quad \text{i.e.} \quad g(\nabla\phi, \nabla f + i\nabla\rho) = 0, \tag{49}$$

and each of ϕ, f, ρ satisfy the system

$$\triangle_g \kappa = 0, \quad (\nabla\kappa)^2 = 0, \tag{50}$$

has the form

$$F = f + \rho[\vec{i_1}G(\phi) + \vec{i_2}H(\phi) + \vec{i_3}P(\phi)] \tag{51}$$

where G, H, P are real valued functions satisfying

$$P^2 + H^2 + G^2 = 1. \tag{52}$$

Thus, F is a section of a $R \times R \times S^2$-bundle over (M, g), where S^2 denotes the two-dimensional sphere. [15]

Proof: Let $F = f + \vec{i_1}k + \vec{i_2}h + \vec{i_3}p \in \mathbf{M}$ be a smooth quaternion-valued function, with $k^2 + h^2 + p^2 \neq 0$. The condition $\triangle_g F = 0$ requires equivalently

$$\triangle_g f = \triangle_g k = \triangle_g h = \triangle_g p = 0. \tag{53}$$

Since by assumption

$$\begin{aligned} (\nabla F)^2 &= (\nabla f)^2 - (\nabla k)^2 - (\nabla h)^2 - (\nabla h)^2 \\ &+ 2\vec{i_1}g(\nabla f, \nabla k) + 2\vec{i_2}g(\nabla f, \nabla h) + 2\vec{i_3}g(\nabla f, \nabla p) = 0 \end{aligned} \tag{54}$$

then it follows that

$$(\nabla f)^2 = (\nabla k)^2 + (\nabla h)^2 + (\nabla p)^2. \tag{55}$$

and

$$g(\nabla f, \nabla k) = g(\nabla f, \nabla h) = g(\nabla f, \nabla p) = 0. \tag{56}$$

The eigenvalues of a quaternion, namely the real or complex numbers λ such that $f + \vec{i_1}k + \vec{i_2}h + \vec{i_3}\rho - \lambda$ is not invertible, or equivalently such that

$$(f - \lambda)^2 + k^2 + h^2 + p^2 = 0 \tag{57}$$

are obviously of the form

$$\lambda_{\pm} = f \pm i\rho, \tag{58}$$

where

$$\rho = (k^2 + h^2 + p^2)^{\frac{1}{2}} > 0, \tag{59}$$

where we remark that i is the commutative square root of minus 1, of complex numbers. It is easy to check that from our previous analysis it follows that

$$\lambda_{\pm} \in \mathbf{M}, \tag{60}$$

[15] We have constructed the quaternions in terms of logical operators in matrix logic [13]. So we can represent this result as an 'objective' space representation of the objective-subjective photon (the seeing process), or -inclusively (surmounting dualism) - as a 'subjective' representation of it in terms of a quaternionic structure which stems from the laws of thought.

which implies that in to addition to eq. (102) we have

$$\begin{aligned} \triangle_g \rho &= 0, &(61)\\ (\nabla f)^2 &= (\nabla \rho)^2 &(62)\end{aligned}$$

$$g(\nabla f, \nabla \rho) = 0, \tag{63}$$

as one obtains from specializing eqs. $(53, 55, 56)$ to the complex case. We now consider the algebra over the reals generated by F (we have to restrict ourselves to real coefficients because the real quaternions $\mathbf{H}$ is a division algebra over $\mathbf{R}$, but over $\mathbf{C}$ it is not).

The analytic functions in the complex plane generated by polynomial with real coefficients are those whose domain is symmetric about the real axis and which satisfy $\Phi(z) = \bar{\Phi}(\bar{z})$ (called intrinsic functions on $\mathbf{C}$ [20]). If $\Phi(x+iy) = u(x,y)+iv(x,y)$ is an intrinsic entire function, and u and v are its real and complex part, respectively, then if x_0, x_1, x_2, x_3 are real, then we have the decomposition of the form

$$\Phi(x_0 + \vec{i_1}x_2 + \vec{i_2}x_2 + \vec{i_3}x_3) = u(x_0, q) + Jv(x_0, q), \tag{64}$$

where

$$q = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \tag{65}$$

and

$$J = \vec{i_1}\frac{x_1}{q} + \vec{i_2}\frac{x_2}{q} + \vec{i_3}\frac{x_3}{q}. \tag{66}$$

This follows directly from the observation that in $x_0 + Jq$, the powers of J behave alike those of i, i.e. $J^2 = -1, J^3 = -J, \ldots$, plus the fact that the coefficients of Φ are real. This is also shown in Theorem. 7.1, page 14 of [20], as a consequence of the fact that the intrinsic functions over $\mathbf{H}$ may be characterized as those that are invariant under the automorphisms or antiautomorphisms ([73]) of $\mathbf{H}$. Using eq. (64) we get

$$\Phi(f + \vec{i_1}k + \vec{i_2}h + \vec{i_3}\rho) = u(f, \rho) + v(f, \rho)(\vec{i_1}K + \vec{i_2}H + \vec{i_3}P) \tag{67}$$

where we have set

$$K = \frac{k}{\rho}, H = \frac{h}{\rho}, P = \frac{p}{\rho}. \tag{68}$$

We now show that

$$\triangle_g K = \triangle_g H = \triangle_g P = 0, \tag{69}$$

and

$$g(\nabla K, \nabla K) + g(\nabla H, \nabla H) + g(\nabla P, \nabla P) = 0. \tag{70}$$

We start by noticing that from eqs. $(56, 63)$ follows that

$$g(\nabla f, \nabla K) = g(\nabla f, \nabla H) = g(\nabla f, \nabla P) = 0. \tag{71}$$

Now

$$\triangle_g \Phi(F) = 0, \tag{72}$$

implies that

$$\triangle_g (vK) = \triangle_g (vH) = \triangle_g (vP) = 0. \tag{73}$$

Since $\nabla v = (\partial_x v)\nabla f + (\partial_y v)\nabla \rho$ and so

$$\begin{aligned} \triangle_g v &= \mathrm{div}\nabla v = (\partial_x v)\triangle_g f + (\partial_y v)\triangle_g \rho \\ &+ (\partial_{xx} v)(\nabla f)^2 + (\partial_{yy} v)(\nabla \rho)^2 + 2(\partial_{xy} v) g(\nabla f, \nabla \rho) \\ &= (\partial_{xx} v + \partial_{yy} v)(\nabla f)^2 = 0, \end{aligned} \tag{74}$$

where we used eqs.$(53, 61, 62, 63)$ and the fact that v is harmonic in (x, y) . Therefore,

$$\begin{aligned} 0 &= \triangle_g (vK) + 2g(\nabla v, \nabla K) + K\triangle_g v \\ &= v\triangle_g K + 2\partial_x v g(\nabla f, \nabla K) + 2\partial_y v g(\nabla \rho, \nabla K) \\ &= v\triangle_g K + 2\partial_y v g(\nabla \rho, \nabla K) \end{aligned} \tag{75}$$

by eq. (71). Since we have v at our disposal, this equation implies that

$$\triangle_g K = 0, \quad g(\nabla K, \nabla \rho) = 0. \tag{76}$$

Indeed, take the intrinsic function

$$\Phi(x + iy) = e^{\lambda(x+iy)} = e^{\lambda x} + ie^{\lambda x}\mathrm{sin}(\lambda y), \lambda \in R. \tag{77}$$

Then, $v = e^{\lambda x}\mathrm{sin}(\lambda y)$ and eq. (74) becomes

$$(\triangle_g K)\mathrm{sin}(\lambda \rho) = -2\lambda g(\nabla \rho, \nabla K)\mathrm{cos}(\lambda \rho). \tag{78}$$

Since $\rho \neq 0$ we can choose $\lambda \neq 0$ so that $\cos(\lambda\rho) = 1$, which gives $g(\nabla K, \nabla\rho) = 0$, and then $\sin(\lambda\rho) = 1$ which gives $\triangle_g K = 0$. Similarly for H and P, so that eq. (69) is proved, together with

$$g(\nabla\rho, \nabla K) = g(\nabla\rho, \nabla H) = g(\nabla\rho, \nabla P) = 0. \tag{79}$$

To prove eq. (68) we first apply eq.(55) to the function in eq.(67),obtaining

$$(\nabla u)^2 = (\nabla(vk))^2 + (\nabla(vH))^2 + (\nabla(vP))^2. \tag{80}$$

Now in view of eqs. $(62, 63)$

$$(\nabla u)^2 = (\partial_x u)\nabla f + (\partial_y u)\nabla\rho)^2 = ((\partial_x u)^2 + (\partial_y u)^2)(\nabla f)^2. \tag{81}$$

Similarly

$$\begin{aligned}(\nabla(vK))^2 &= v^2(\nabla K)^2 + 2vKg(\nabla v, \nabla K) + K^2(\nabla v)^2 \\ &= v^2(\nabla K)^2 + K^2((\partial_x v)^2 + (\partial_y v)^2)(\nabla f)^2,\end{aligned} \tag{82}$$

because $g(\nabla v, \nabla K) = 0$ in view of eqs.$(71, 79)$. Analogous expressions for H and P hold, since $(\partial_x u)^2 + (\partial_y u)^2 = (\partial_x v)^2 + (\partial_y v)^2$ by the Cauchy-Riemann equations, and then eq. (80) becomes

$$\begin{aligned}((\partial_x u)^2 + (\partial_y u)^2)(\nabla f)^2 &= v^2[(\nabla)^2 K + (\nabla H)^2 + (\nabla P)^2] \\ &+ ((\partial_x u)^2 + (\partial_y u)^2))(\nabla f)^2,\end{aligned} \tag{83}$$

because $K^2 + H^2 + P^2 = 1$. This gives eq. (70). Therefore we have proved that K, H and P all belong to $\mathbf{M}$.

We now show that there is a real valued function ϕ such that K, H and P are functions of ϕ. For this purpose we will show necessarily that

$$\begin{aligned}(\nabla K)^2 &= (\nabla H)^2 = (\nabla P)^2 = 0, \qquad (84)\\ g(\nabla K, \nabla H) &= g(\nabla K, \nabla P) = g(\nabla H, \nabla P) = 0 \qquad (85)\end{aligned}$$

everywhere. First we note that the group of automorphisms and antiautomorphisms of $\mathbf{H}$, which are precisely the rotations that leave the real unit $1 = 1 + 0\vec{i_1} + 0\vec{i_2} + 0\vec{i_3}$, possibly combined with reflections preserve eqs. $(53, 55, 56, 59)$ as well as the condition $K^2 + H^2 + P^2 = 1$. Furthermore, the intrinsic functions on the quaternions are invariant under this group. Therefore we may always apply a constant rotation on the space of $\vec{i_1}, \vec{i_2}, \vec{i_3}$ to make K, H, P not zero at a particular point $p \in M$. Then it is clear that the new H, K, P will satisfy eqs.

(84, 85) if and only if the original ones they are satisfied by the original functions. Suppose that this is the case at p. Since $P = (1 - H^2 - K^2)^{\frac{1}{2}} > 0$ at p, and so they satisfy it in a neighbourhood of p, and $\triangle_g P = 0$, we get by differentiation

$$2(1 - K^2 - H^2)(-\triangle_g K^2 - \triangle_g H^2) = (\nabla(K^2 + H^2))^2, \tag{86}$$

i.e.

$$\begin{aligned} 2 \quad & (K^2 + H^2 - 1)[2K\triangle_g K + 2(\nabla K)^2 + 2H\triangle_g H + 2(\nabla H)^2] \\ = \quad & 4K^2(\nabla K)^2 + 4H^2(\nabla H)^2 + 8HKg(\nabla H, \nabla K). \end{aligned} \tag{87}$$

Using eq. (53) and simplifying

$$(1 - K^2)(\nabla H)^2 + (1 - H^2)(\nabla K)^2 = -2HKg(\nabla H, \nabla K). \tag{88}$$

Now if ∇H and ∇K are space-like [16], Schwarz's inequality

$$|g(\nabla K, \nabla H))| \leq |\nabla K|^2 |\nabla H|^2 \tag{89}$$

applies. Therefore taking absolute values in eq. (88) we get, since $(\nabla H)^2$ and $(\nabla K)^2$ have the same sign and $1 - K^2 > 0$, $1 - H^2 > 0$,

$$(1 - K^2)|\nabla H|^2 + (1 - H^2)|\nabla K|^2 \leq 2|HK||\nabla K||\nabla H|, \tag{90}$$

i.e.

$$(1 - K^2)|\nabla H|^2 - 2|HK||\nabla H||\nabla K| + (1 - H^2)|\nabla K|^2 \leq 0. \tag{91}$$

The determinant of the matrix of this quadratic form in $(|\nabla K|, |\nabla H|)$ is $(1 - K^2)(1 - H^2) - K^2H^2$= 1-K$^2H^2 - P^2 > 0$ and its trace is

$$2 - K^2 - H^2 - P^2 > 1. \tag{92}$$

so its eigenvalues are positive. This implies in eq. (91) that $|\nabla H| = |\nabla K| = 0$, i.e.

$$(\nabla K)^2 = (\nabla H)^2 = 0, \tag{93}$$

and so also

$$(\nabla P)^2 = 0, \tag{94}$$

[16]For this condition the case of g being positive-definite is automatically satisfied, while in the Lorentzian case it has to be assumed.

by eq.(70) and

$$g(\nabla H, \nabla K) = 0 \tag{95}$$

by eq. (90). Interchanging the roles of H, K, P in eq. (90) we get now the remaining equations in eq. (85).

Therefore, eqs. $(84, 85)$ hold when any two of the vectors $\nabla K, \nabla H, \nabla P$ are space-like (as we said, for g Riemannian this is always the case), since this property is preserved under the small rotation that may be needed to make $H, K, P \neq 0$ at a given point $p \in M$.

In the Lorentzian case, we are left to consider the case when just one of them is space-like, say ∇H, one is time-like, say ∇P, and the third one ∇K is time-like or isotropic. Now the small rotation that may be necessary to make $H, K, P \neq 0$ at a given $p \in M$, may change the character of ∇K if it is isotropic. If it becomes space-like, we are back into the previous case, so we need consider only the remaining case whenever M is Lorentzian (for the Riemannian case, this case is empty).

Clearly then the subspace determined by ∇K and ∇H cuts the light-cone and so we may rotate ∇H and ∇K by the above procedure till ∇H cuts the light-cone at the point p, while leaving P and ∇P unchanged. Since ∇H becomes isotropic, ∇K must then get space-like so as to compensate $(\nabla P)^2$ in eq. (85). Therefore, by continuity, just before ∇H touches the light-cone at the point p, both ∇K and ∇H will be space-like and since $P \neq 0$, this then reduces the problem to the previous case. This proves that eqs. $(84, 85)$ hold everywhere.

To complete the proof we only need observe that any two real isotropic vectors which are orthogonal in the Lorentzian manifold (M, g), are necessarily parallel. Hence if $\nabla H \neq 0$, necessarily $\nabla P = \mu \nabla H, \nabla K = \lambda \nabla P$ with μ, λ real functions, and therefore $P = P(H), K = K(H)$ locally, as claimed. In view of eqs.$(71, 79, 84)$, we conclude that

$$(H, f + i\rho) \in \mathbf{M} \tag{96}$$

concluding thus with the proof of Theorem 1.

4.2 Maximal Monochromatic Algebras

A monochromatic algebra is called maximal monochromatic if it is not a proper subalgebra of a monochromatic algebra. The importance of maximal monochromatic algebras in our context is obvious, in particular with respect to the question of singular sets. The main result in this respect is

Theorem 2. The maximal C^2 algebras in (M, g) are precisely those generated by a single pair (see (49,50))

$$(\phi, f + i\rho) \in \mathbf{M}, \tag{97}$$

with ϕ, f, ρ real, and are C^2 functions of the form

$$\xi(f, \phi, \rho) + \eta(f, \phi, \rho)[\vec{i_1}K(\phi) + \vec{i_2}H(\phi) + \vec{i_3}P(\phi)] \tag{98}$$

in the quaternionic case, and

$$\xi(f, \rho, \phi) + i\eta(f, \rho, \phi), \tag{99}$$

in the complex case, where for each fixed ϕ, $\xi + i\eta$ is an intrinsic analytic (or antianalytic [17]) function of $f+i\rho$, of class C^2 on ϕ is arbitrary and $K^2+H^2+P^2 = 1$,with K, H, P of class C^2, but otherwise arbitrary. Thus, in the quaternionic case, it is given by a C^2-section of a $R \times R \times S^2$-bundle over M. In the complex case, non intrinsic functions are allowed.

Proof. We first prove that the most general quaternionic valued monochromatic of class C^2 function of a pair $(\phi, f + i\rho) \in \mathbf{M}$ has the form (89). Indeed, let $F(f, \rho, \phi) \in \mathbf{M}$ be a continuously differentiable up to order two quaternionic function. By Theorem 1, it has the expression

$$F = \xi + \eta[\vec{i_1}\Gamma_1(\Phi) + \vec{i_2}\Gamma_2(\Phi) + \vec{i_3}\Gamma_3(\Phi)] \tag{100}$$

with $(\Phi, \xi + i\eta)) \in M, \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = 1, \Gamma_i$ real valued.

By assumption ξ, η and Φ are functions of f, ρ, ϕ . Since $\xi + i\eta \in \mathbf{M}$ and $\Phi \in \mathbf{M}$, it suffices therefore that we analyze the problem for these particular functions, and this reduces the problem to the case when F is a real or complex-valued function of the pair $(\phi, f + i\rho) \in \mathbf{M}$. Now, using the properties of this pair we get from $\nabla F = F_\phi \nabla\phi + F_f \nabla f + F_\rho \nabla\rho$, that

$$(\nabla F)^2 = (F_f^2 + F_\rho^2)(\nabla f)^2. \tag{101}$$

As f is independent of ϕ, then $(\nabla f)^2 \neq 0$ as remarked earlier. Therefore, necessarily

$$F_f^2 + F_\rho^2 = (F_f + iF_\rho)(F_f - iF_\rho) = 0, \tag{102}$$

[17] A function defined on an open set in the complex plane is called antianalytic (or antiholomorphic) if its derivative with respect to $\bar{z}$ exists at all points in that set, where $\bar{z}$ is the complex conjugate.

i.e. F must be analytic or anti-analytic function of $f+i\rho$. No additional restriction is placed on F as a function of ϕ. The condition $\nabla_g F = 0$ is automatically satisfied since

$$\begin{aligned}
\triangle_g F &= F_\phi \triangle_g \phi + F_\rho \triangle_g \rho + F_f \triangle_g f + 2F_{\phi\rho} g(\nabla\phi, \nabla\rho) \\
&+ 2F_{\rho f} g(\nabla f, \nabla\rho) + 2F_{f\phi} g(\nabla f, \nabla\phi) + F_{\phi\phi}(\nabla\phi)^2 \\
&+ F_{\rho\rho}(\nabla\rho)^2 + F_{ff}(\nabla f)^2 \\
&= (F_{ff} + F_{\rho\rho})(\nabla f)^2 = 0,
\end{aligned} \tag{103}$$

as F is harmonic in (f,ρ).

Therefore $\xi(f,\rho,\phi)$ and $\eta(f,\rho,\phi)$ satisfy the stated conditions, and so do the Γ_is. Since Γ_i are real-valued they are therefore constant on $f+i\rho$, i.e. they depend on ϕ only.

The complex case is obtained by specializing $H \equiv P \equiv 0$, $K \equiv 1$, and by complexification, non-intrinsic functions are obtained.

It is easy to check that eq. (98) belongs to the real algebra generated by ϕ and $f+i\rho$ (or, on ϕ and $f-i\rho$ if it anti-analytic in $f+i\rho$). Similarly, eq. (99) belongs to the complex algebra generated by ϕ and $f-i\rho$, as before). The same applies trivially to functions of ϕ only.

We finally prove the maximality condition. In any of the two cases above let a monochromatic algebra contain $(\phi, f+i\rho) \in \mathbf{M}$ and a third function F. By Theorem 1 this function is given in terms of a pair $(\tilde\phi, \tilde f + i\tilde\rho) \in \mathbf{M}$. If the function is trivial, it is expressible as a function of $(\phi, f+i\rho)$ too. If not, it depends non-trivially on at least one of $\tilde\phi, \tilde f + i\tilde\rho$. In that case, since the functions belong to a monochromatic algebra, the corresponding $\nabla\tilde\phi$ and/or $\nabla(\tilde f + i\tilde\rho)$ must be orthogonal to both $\nabla\phi$ and $\nabla(f+i\rho)$. (Notice that the latter commute with ∇F , in the scalar product).

If $g(\nabla\tilde\phi, \nabla\phi) = 0$ locally, then necessarily $\tilde\phi = \tilde\phi(\phi)$ as both are real monochromatic and so $\tilde\phi$ belongs to the algebra of ϕ.

If $g(\nabla(\tilde f + i\tilde\rho), \nabla(f+i\rho)) = 0$ and $g(\nabla(\tilde f + i\tilde\rho), \nabla\phi) = 0$, then we have

$$\nabla(\tilde f + i\tilde\rho) = \alpha(x)\nabla\phi + \beta(x)\nabla(f+i\rho), \tag{104}$$

with α, β complex-valued functions defined on M by the Lemma below. This implies that $\tilde f + i\tilde\rho \in \mathbf{M}$ is (locally) a function of (f,ρ,ϕ) and so by Theorem 1, belongs to the algebra generated by $(\phi, f+i\rho)$.

Therefore, Theorem 2 is proved once we prove the following Lemma, valid only for g a Lorentzian metric (i.e. only the degenerate metric case).

Lemma. If two isotropic vectors [18] v_1, v_2 are orthogonal to a real isotropic vector v_3 in Minkowski space, then either v_1, v_2, v_3 or $v_1, \tilde{v}_2, v_3$ are linearly dependent. If v_2 is orthogonal to v_1 then the first case holds.

Proof. We may assume that the real vector is $(1, 1, 0, 0)$, the signature being $\mathrm{diag}(-1, 1, 1, 1)$. Since linear combinations of v_1, v_2 with v_3 preserve their stated properties we can make the first components of v_1, v_2 into zero by adding a convenient multiple of v_3. But then since they are orthogonal to the real vector also their second components are zero. So they are of the form $(0, 0, a, b), (0, 0, c, d)$ and by isotropy $a^2 + b^2 = c^2 + d^2 = 0$, i.e. $b = \pm ia, d = \pm ic$. Hence they are multiples of $(0, 0, \pm i, 1)$ and $(0, 0, 1, \pm i)$. For any choice of sign, these vectors are equal or one is equal to the complex conjugate of the other. They can be orthogonal only in the first case. The result follows.

Clearly the result holds pointwise for an arbitrary Lorentzian manifold (M, g) since we can always make $g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ at a fixed point p.

Remarks.1. In view of Theorem 2, all functions of $f + i\rho$ in the same algebra must be simultaneously analytic or anti-analytic in the same connected regions. For simplicity we refer to them as analytic bearing in mind these two possibilities.
2. It is clear that $(\phi, f + i\rho) \in \mathbf{M}$ implies that $(\phi, f - i\rho) \in \mathbf{M}$. However since the analytic functions in $f - i\rho$ are precisely the antianalytic functions in $f + i\rho$ and viceversa, we will not consider these two pairs as distinct, because they generate the same maximal monochromatic algebras, according to Theorem 2.

5 General Form of Singular Sets, and Their Physical Interpretations

We are now in conditions for completing the objective of the previous Section, namely, the characterization of the node set of complex and quaternionic monochromatic functions. According to the above results the most general form for singular sets N of monochromatic complex or quaternionic functions is given by the conditions

$$\begin{aligned} \xi(f, \rho, \phi) &= 0, \\ \eta(f, \rho, \phi) &= 0, \quad (\phi, f + i\rho) \in \mathbf{M} \end{aligned} \tag{105}$$

Although N is locally at least two-dimensional we have now the possibility of locating a higher-order zero on a bicharacteristic line.

[18] Later we shall name them as *null vectors*, i.e. their length is 0.

For instance, the singular set of $\phi.(f + i\rho)$ is the union of the 2-dimensional set defined by $f = \rho = 0$, and the 3-dimensional set $\phi = 0$, and since $(\nabla\phi)^2 = 0$, their intersection $f = \rho = \phi = 0$ is a bicharacteristic line carrying an isolated zero of higher order. The corresponding phase function has a higher order singularity located at a single point in three-space, moving with the speed of light along the singular line $f = \rho = 0$, accompanied by the wave-front singularity $\phi = 0$.

Observations. This result is remarkable in many ways. Firstly, in the present analytic approach, it is apparent that the photon cannot exist per-se as a point-like singularity, since in fact the most general maximal monochromatic algebra is three-dimensional when we go to the quaternionic case and then we can describe it as built in the larger singularity. Thus, we have in a four-dimensional Lorentzian manifold, three real dimensions to describe the eikonal wave discontinuities. This precisely gives a representation for a recent conjecture by Kiehn that the photon is to be regarded as a three-dimensional singularity [23]. Furthermore, in Kiehn's point of view, the photon is associated with a spinor, which are the natural representations for propagating singularities; we shall present another approach to this below. Furthermore, spinors are related to conjugate minimal surfaces [19], which for the photon are described by Falaco solitons [23]. Now, regarding higher dimensional singularities, they always give rise to the de Rham's topological rules. For three dimensional singularities, we have the topological torsion first Poincaré invariant, while for four dimensional singularities, we have the spin-torsion second Poincaré invariant , the Euler number of the manifold. It is apparent from the present theorem, that for the construction of the three dimensional singularities, monochromatic algebras are enough for their generation. These represent thermodynamically irreversible processes which exchange energy but no matter with the environment (the photon), while the four-dimensional highest order case, represent the case in which there is exchange of matter [23]. We shall retake in detail these issues elsewhere.

5.1 On the Intelligence Code and Some Metrics in Cosmology

We recall or initial discusion on Fock's critique to GR [4] as a theory of uniform space, and the need of singularities to establish an 'objective' spacetime. We have discussed already that the primeval distinction that encodes the torsion field is such a singularity [13]. Thus, in analyzing the hyperbolic nature of the Einstein's partial differential equations of GR and the Maxwell equations, Fock was lead to propose as a starting point the eikonal and propagation wave equations,

[19] This is most remarkable since it points out to the existence of a Platonian world, with generic geometrical surfaces associated to the subjective-objective photon.

whose wavefronts correspond precisely to propagating singularities, that we have further associated with torsion. Furthermore, using the functions defining the maximal monochromatic algebra it is possible to establish a coordinate system for spacetime without recourse to an ad-hoc non-geometric energy-momentum tensor as Einstein's inception of it in GR (the "right hand side made of mud", in Einstein's words). This is done using the energy-momentum tensor of the electromagnetic field by solving the Einstein's equations of GR with light as source for the gravitational field described by the curvature derived from the metric in the Levi-Civita connection. This provides a self-referential construction of the metric which is absent in the conception of GR. Further below we shall relate this maximal monochromatic algebra to cognitive states which would thus generate a spacetime metric inverting our common undestanding of the 'exterior' world as being passively represented by the mind but disregarding the inverse direction of constitution of reality as *jointly* operational in a Kliein bottle sense. This understanding is further supported by the fact that the quaternion field $\mathbf{H}$ can be constructed in matrix logic [13]. Now, the natural metric in the Lie group of the invertible quaternions can be parametrized as the closed Friedmann-Lemaitre-Robertson-Walker metrics [75] which constitute one of the most important classes of solutions of Einsteins equations and furthermore, as the Carmeli metric of rotational relativity. We recall that the latter was introduced to explain spiral galaxies rotation curves and 'dark matter' [76]. We stress that these derivations do *not* require solving the Einstein's equations of GR but are intrinsic to $\mathbf{H}$. So the Intelligence Code has some remarkable built-in metrics that are purported to describe cosmological phenomenae.

5.1.1 Photon, Nodal Lines, Monopoles

Until know we have described the singular sets of quaternionic and complex solutions of the eikonal and wave propagation equations. A typical case is to establish a Cartesian coordinate system $(x, y, z, t) \in \mathbf{R}^{1,3}$ (in Minkowski space) given by taking

$$f + i\rho = y + iz, \tag{106}$$

and

$$\phi = f(r) - t, \text{with } \ r = (x^2 + y^2 + z^2)^{\frac{1}{2}}, \tag{107}$$

and f is a monotonic function of the radius r. In this case, for the function

$$(y + iz)(f(r) - t) \tag{108}$$

the singular set consists of a spherical wave front in 3-space moving with the speed of light and cutting the singular x-axis $y = z = 0$ at a single point in the positive semi-axis $0 \leq x$, where therefore lies a higher-order singularity. This higher-order singular point, piloting a lower order singular spherical wave along a lower order singular line, is now liable to represent the photon. Here the photon is conceived as a moving point singularity carrying energy, in agreement with the experimentally observed corpuscular behaviour of the photon at a metallic plate, and obtain pictures of its trajectories in cloud chambers. On the other hand the weaker singularity carried by the spherical wave front $f(r) - t = 0$ is responsible for the diffraction patterns in the typical slit experiments, according to Huygens's law of propagation of singularities (eikonal equation), and so accounts for the experimentally observed wave nature of the photon. In this way the purely analytical characterization of the maximal monochromatic algebras leads us unequivocally, to the correct conclusions as regards the physical nature of the photon and express its *purportedly* dual wave-corpuscular nature as a simple mathematical fact. It is essentially a wave, the particle being a factor of it, but not dual in any intrinsic sense. Remarkably, this stands in contrast with the de Broglie-Vigier double solution theory [81], in which the wavelike pattern is associated to a linear propagation (alike eq. (8)) while the particle was treated as a propagating singularity ascribed to a non-linear equation, which in the present theory is eq. (9). Thus the present theory fleshes out in a completely geometric setting, the double solution theory, which appears in the torsion geometry of the linear and nonlinear Dirac-Hestenes equation [10], (2005), yet it relinquishes duality.

The line $y = z = 0$ in 3-space carries a singularity too, but this is a standing one, independent on time, and therefore, its presence is detected through different effects. Actually this line is so-called a nodal line of the wave function $\phi \equiv y + iz$ ([24]) or a dislocation line of the planes of constant phase of ψ. Around this line occur vortices of the flux of the trace torsion one-form $d\ln\phi$ of the phase function (when the circulation of this flux around a nodal line is non-zero), described in detail in [32]. Alternatively Dirac found these nodal lines when considering singularities of wave functions, upon imposing the only requirement that the complex-valued functions ψ (in our example equal to $y + iz$) be single-valued and smooth, but not necessarily with single-valued argument, and then quantized them in terms of the winding number of the vector-field $(\nabla \ln Re(\psi), \nabla \ln Im(\psi))$ along a closed curve around the line. He then found that one could remove the non-zero circulation by means of a gauge transformation of the second kind, and that the electromagnetic vector potential associated with this transformation was precisely the same electromagnetic potential produced by a magnetic monopole at the initial point. He then equated the effect of the circulation around the nodal line in the original gauge to the effect of a monopole in the new gauge. His quantization by the winding number is actually just a *special case of the general quantization theorem* above, and his gauge interpretation is thus a concrete exemplification of the meaning of the analysis given there.

The variety of types of singular sets defined by the representation given in eqs. (105) is very great, as exemplified in the pioneering work [24]. Besides the singular sets that we previously identified with the photon (spherical wave front plus a nodal line) there is also a remarkable singular set of the monochromatic wave constructed out of $\phi = f(r) - t$ and $f + i\rho = y + iz \in \mathbf{R}^{1,3}$, by the following sum

$$\epsilon e^{i\omega[f(r)-t]} - (y + iz), \epsilon > 0. \tag{109}$$

Its singular set is given by

$$y = \epsilon \mathrm{cos}\omega[f(r) - t], z = \epsilon \mathrm{sen}\omega[f(r) - t]. \tag{110}$$

This represents an helicoidal line lying on the cylinder $y^2 + z^2 = \epsilon^2$, and moving with (variable) speed of light along its tangent direction at each of its points. (For simplicity, we can assume that $f(r) = t$ in order to get a better visualization: the speed is then constant and the helicoid has then a constant step.) Taking $y - iz$ instead, we get a screw motion with opposite handedness. The singular set is thus a moving screw in 3-space that can be right or left handed, and may carry the energy associated with a quantum jump, as shown above. It seems therefore that a monochromatic wave line like this can represent appropiately a right or left handed neutrino, concretely identified with its singular set. It has then quite distinct properties from those associated with a photon. For it is given by an infinitely long moving right of left handed helicoidal line in 3-space (which by the way, it is a minimal surface; more on dislocations and minimal surfaces and turbulence, shall be presented elsewhere) while the photon is given by a point piloting a spherical wave. In particular if the singular screw line of above is associated with an elementary state u and carries energy E in the manner described in [61] it also carries the angular momentum $\epsilon^2 E\omega/c^2$ directed along a x-axis, in the given referential. Hence the neutrino carries angular momentum while the photon does not. On the other hand, according to this description, the neutrino should not have (primary) diffraction patterns as the photon does, which should explain why it is so difficult of detect. The infinitely long screw line seems to agree, in principle, with the experimentally estimated fact that the neutrino has an extremely long absorption path.

5.1.2 Distinction of Maximal Monochromatic Algebras

Lemma 2. The maximal monochromatic algebras $\mathbf{M_1}$ and $\mathbf{M_2}$ with generators $(f, \phi + i\psi)$ and $(\tilde{f}, \tilde{\phi} + i\tilde{\psi})$ respectively, are distinct if and only if

$$g(\nabla f, \nabla \tilde{f}) = 0. \tag{111}$$

Proof. If $g(\nabla f, \nabla \tilde{f}) = 0$, then necessarily f is a function of $\tilde{f}$ and therefore

$$g(\nabla f, \nabla \tilde{\phi} + i\nabla \tilde{\psi}) = 0, \tag{112}$$

besides

$$g(\nabla f, \nabla \phi + i\nabla \psi) = 0. \tag{113}$$

But then from Lemma 1 follows that either ∇f, $\nabla\phi + i\nabla\psi$, $\nabla\tilde{\phi} + i\nabla\tilde{\psi}$ or $\nabla f, \nabla\phi + i\nabla\psi$ and $\nabla\tilde{\phi} - i\nabla\tilde{\psi}$ are linearly dependent. In any case, $\tilde{\phi} + i\tilde{\psi}$ is a function of $(f, \phi + i\psi)$ and so is $\tilde{f}$, the two algebras are the same (here again we are considering the possibility of having to consider analytic functions in one and anti-analytic in the other).

Conversely, assume the algebras are not distinct. Then by Theorem 2, $\tilde{f}$ is a function of $(f, \phi + i\psi)$, analytic or anti-analytic in $\phi + i\psi$, and then $g(\nabla f, \nabla \tilde{f}) = 0$. c.q.d.

5.2 Spinor and Twistor Description of Maximal Chromatic Algebras

If (x^0, x^2, x^2, x^3) is a vector in Minkowski space $\mathbf{R}^{1,3}$ we may associate with it the 2×2 hermitean matrix

$$x = (x^0, x^1, x^2, x^3)X = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{pmatrix} \tag{114}$$

This is obviously a linear isomorphism of $\mathbf{R}^{1,3}$ with the real space of 2 times 2 hermitean matrices. Direct computation shows then that when the vector is acted upon by a proper Lorentz transformation L, the associated hermitean matrix undergoes multiplication by a 2 times 2 complex unimodular matrix on the left and by its transpose conjugate on the right. The unimodular matrix is uniquely determined by L, except for sign of course. This correspondence gives an isomorphism between the group $Sl(2, \mathbf{C})$ of 2 by 2 unimodular complex matrices and the two-fold universal covering group of the connected subgroup of the Lorentz group $O(1,3)$. A spin vector on (M,g) is defined locally by taking a smooth moving orthonormal frame, i.e., such that on each point the metric has the standard form $(dx^0)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2$, and assigning, in a smooth way, a spin vector at each point of the corresponding tangent manifold. Since in the above correspondence we have

$$(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2=\frac{1}{2}\det(X), \tag{115}$$

then any (real) isotropic vector (i.e. a non-zero vector whose Minkowski length is 0, also called a null vector, or still, a Cartan spinor) corresponds to a hermitean matrix given by the tensor product of a spin vector ω^A with its complex conjugate $\bar{\omega}^{A'}$ (the vector is then called future-pointing) or with $-\bar{\omega}^{A'}$ (the vector is called past pointing), $(A,A'=1,2)$.

The spin vector ω^A is determined by the vector, up to a factor $e^{i\theta},\theta$ real, obviously. This extra degree of freedom relates to a possible polarization of the objects involved. Clearly complex isotropic vectors are given by the tensor product $\omega^A\pi^{A'}$ of spin vectors. Finally we remark that two isotropic vectors are orthogonal if and only if either the first or their second associated spin vectors are parallel. Indeed, let in matrix form

$$u=\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c & d \end{pmatrix}, v=\begin{pmatrix} e \\ f \end{pmatrix}\begin{pmatrix} g & h \end{pmatrix} \tag{116}$$

Using polarization of bilinear forms and eq. (115) we get

$$\begin{aligned}
u^iv_i &= \frac{1}{4}[(u_i+iv_i)(u^i+v^i)-(u^i-v^i)(u_i-v_i) \\
&= \frac{1}{4}\det[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c & d \end{pmatrix}+\begin{pmatrix} e \\ f \end{pmatrix}\begin{pmatrix} g & h \end{pmatrix}] \\
&- \frac{1}{4}\det[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c & d \end{pmatrix}-\begin{pmatrix} e \\ f \end{pmatrix}\begin{pmatrix} g & h \end{pmatrix}] \\
&= \frac{1}{2}(af-be)(ch-dg).
\end{aligned} \tag{117}$$

Hence, $u^iv_i=0$ if and only if $af=be$ or $ch=dg$, as claimed.

5.2.1 Twistors and Maximal Monochromatic Algebras

Consider the generators $(\phi,f+i\rho)\in\mathbf{M}$ of a maximal monochromatic algebras. The vectorfields $\nabla\phi$ and $\nabla(f+i\rho)$ are, respectively, real and complex isotropic fields, mutually orthogonal on (M, g); here M is a generic spin-manifold provided with a Lorentzian metric g. By the previous analysis $\nabla\phi$ is given by a spin vectorfield ω^A in the spinor form

$$\nabla\phi = \omega^A \bar{\omega}^{A'} \quad (\text{or} - \omega^A \bar{\omega}^{A'}). \tag{118}$$

and since $\nabla(f + i\rho)$ is isotropic and orthogonal to $\nabla\phi$ then we have

$$\nabla(f + i\rho) = \omega^A \bar{\pi}^{A'}, \tag{119}$$

where π^A is another spin vectorfield. Consequently the pair $(\nabla\phi, \nabla(f + i\rho))$ of vectorfields is completely determined by the ordered pair of spin vectorfields

$$(\omega^A, \pi^{A'}), \tag{120}$$

but we have a fourfold map here since we have already altogether four different ways of building the vectorfields according to eq. $(118, 119)$ out of the ordered pair (120). The correspondence (118), extended to complex vectors x shows that the second choice in (147) reverses $\nabla\phi$ from, say , a future-pointing isotropic vector to a past-pointing isotropic vector while in (118) it chooses the complex-conjugate $\nabla(f - i\rho)$ instead of $\nabla(f + i\rho)$, reversing the roles of analytic and anti-analytic functions, which means inversion of handedness. Choosing locally a given time orientation and a given handedness, corresponds to a particular choice of the assignements in eqs. (118,119). The ordered pair (120) of spin vectorfields at a point in (M, g) is called a local twistor and the corresponding field a local twistor field. From the twistor field we determine the real and complex vectorfields by eqs. (118, 119,114), which, upon integration yield an equivalent pair $(\phi, f + i\rho)$. This means that we can characterize completely a maximal monochromatic algebra (and consequently the light quanta it represents) in terms of a twistor field with divergence free associated vectorfields.[20]. The new representation is even richer as it has built in an extra degree of freedom, namely, polarization, due to the

[20] This is remarkable in relation to the fact that the Intelligence Code which we shall present below is a *nilpotent universal rewrite system* in the sense of [15] albeit generated by the Klein bottle, yet due to the representation by Musès hypernumbers of logical operators, non-trivial square roots of $+1$ in addition of those of -1 appear; the latter is the case in [15]. We recall that photons appear as propagating singularities for both the eikonal and the Maxwell equations [4]. The latter equations, under certain conditions -that amount to reduce 4 to 2 degrees of freedom (in the Dirac algebra)-, similarly as in the (Gupta-Bleuler) quantization of the electromagnetic field, are equivalent to the Dirac-Hestenes linear and non-linear equation of relativistic quantum mechanics [10] (2005). In spite that we enlarged the original complex fields to quaternion-valued ones to derive the characterization of the extended photons by a maximal monochromatic algebra and its spinor and twistor descriptions, we showed above that it only requires the canonical

factor $e^{i\phi}$mentioned before. This result, showing that we have identified as light quanta are indeed given by twistor fields, substantiates the belief of Penrose that twistors are the appropiate tool to describe zero rest-mass particles and to effect the connection between gravitation and quantum mechanics, and particularly, Kiehn's conjecture [23]. Further below we shall see that this connection extends to the laws of thought.

5.3 Classical Interpretation, Helicity and Spin

We follow the original definition of Penrose of twistors in Minkowski space, which starts with the fact that if a zero rest-mass particle has linear momentum p^a ($a = 0, 1, 2, 3$) and angular momentum M^{ab} with $M^{ab} = -M^{ba}$ ($a, b = 0, 1, 2, 3,$) with respect to some point taken as an origin, then we can write them in spinor form as

$$p_{AA} = \bar{\pi}_A \pi_{A'}, M^{AA'BB'} = i\omega^{(A}\bar{\pi}^{B)}\epsilon^{A'B'} - i\epsilon^{AB}\bar{\omega}^{(A'}\bar{\pi}^{B')}, \tag{121}$$

where brackets stand for symmetrization, $Z^\alpha = (\omega^A, \pi_{A'})$ is a twistor, and

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{122}$$

is the spinor index raising operator; further below in eq. (144) we shall re-encounter is as a TIME operator in logic. We recall that the twistor is only defined up to a real phase change $e^{i\theta}$, with $\theta \in \mathbf{R}$. The vector p_a is an eigenvector of M^{ab}, i.e.

$$\frac{1}{2} e_{abcd} p^b M^{cd} = s p_a, \tag{123}$$

where $e_{0123} = 1$ and e_{abcd} is antisymmetric in all indices. The eigenvalue s is the helicity of the particle and $|s|$ its spin. It is also given by

$$2s = Z^\alpha \bar{Z}_\alpha = \omega^A \bar{\pi}_A + \pi_{A'} \bar{\omega}^{A'}, \tag{124}$$

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commutative square root of -1 of complex numbers (whose matrix representation will appear to be the TIME operator in matrix logic). This is also the case of the *infinite*-dimensional representations of spinors and the Dirac equation for them [62]. This explicit independence of the *non*-commutative square roots of -1 will carry out to the codification of the mind apeiron in terms of twistors, and thus at the primeval zero observable of thought (which we called the mind apeiron) they play no role. As well known, the non-commutative square roots of -1 naturally only show up for the *finite* dimensional Dirac algebra which is the one used in [15].

where the second term stands for the scalar product of the twistor Z^{α} with its complex conjugate $\bar{Z}_{\alpha} = (\bar{\pi}_A, \bar{\omega}^{A'})$. Let the twistor $Z^{\alpha} \neq 0$ be null, i.e. $Z^{\alpha}\bar{Z}_{\alpha} = 0$, then there is a single line Z of points with respect to which $M^{ab} = 0$, and it is parallel to p^a, and therefore isotropic. If $X^{\alpha} \neq 0$ is another null twistor, with isotropic line X, then X and Z meet if and only if $X^{\alpha}\bar{Z}_{\alpha} = 0$ [22]. The isotropic line Z describing the twistor Z^{α} up to a factor, is thus completely characterized by the congruence given by the isotropic line that meet Z, i.e. by the set

$$\{X | X^{\alpha}\bar{X}_{\alpha} = 0, X^{\alpha}\bar{Z}_{\alpha} = 0\} \tag{125}$$

If $Z^{\alpha}\bar{Z}_{\alpha} \neq 0$ we can again describe Z^{α} by the congruence of isotropic lines that satisfy the previous equation but now there is no isotropic line assoicated with Z^{α} and the lines associated with X^{α} twist about one another (right-handedly when $s > 0$, left-handedly when $s < 0$) and never intersect.

6 Self-reference, the Klein Bottle, Torsion, the Laws of Thought and the Twistor Structure of the Cognitive Plenum

6.1 Introduction

Up to know we have elaborated a theory of torsion and photons, which we departed presenting it as a theory of an 'objective' realm that has its standing in the Cartesian cut mindset. This mindest corresponds to a world in which subjectivity does not participate, or altogether does not exist in the universe of discourse. Yet, we could not keep this cut, having shown that both torsion and the photon are very closely related to self-reference, and thus to consciousness [13]. Furthermore, the semiotic codification of torsion as a distinction sign produces in incorporating paradox, a multivalued logic which is associated with the Klein bottle and time waves [13]. From this logic, it was proved that the most general matrix-tensor logic that has as particular cases quantum, fuzzy, modal and Boolean logics [26] stem from these time waves. In this theory which stems from abandoning the scalar logic theory of Aristotle and Boole, promoting it to logical operators, we find that the Klein bottle plays a fundamental role as an in-formation operator, which coincides with the Hadamard gate of quantum computation. The role of this gate is to transform the vector Boolean states to superposed states, the latter being associated with the torsion of cognitive space and the non-orientability of this space due to its constitution in terms of the self-referential non-orientable Moebius and Klein bottle surfaces. Furthermore, the logical cognitive operator which leads to quantization of cognition, is generated by the torsion produced from the commutator of the FALSE and TRUE logical operators which self-referentially involutes to give the difference between these two operators, as we shall see below. The picture that stems is that matrix logic can be seen as the self-referential logical code which stands at the foundation of quantum physics to which is indisolubly related. We have elaborated the relations between matrix logic, self-reference, non-orientability and the Klein bottle, nilpotent hypernumber representations of quantum fields that represent some logical operators [13]. Thus, in this theory, matter quantum field theories are logical operators, and viceversa, and a transformation between quantum and cognitive logical observables has been established. This theory has produced a new fundamental approach to the so-called mind-matter problem , establishing its non-separateness, and the primacy of consciousness which thus *cannot* be claimed to be an epiphenomenon of physical or other complex fields [13]. By promoting the 'truth tables' of usual Boolean logic to matrix representations, the founder of matrix logic, A. Stern, was able to produce an operator logic theory in which logical operators may admit inverses, and the operations of commutation and anticommutation are natural [26]. Furthermore, logical operators can interact by multiplication or addition and, in some cases,being invertible, they yield thus to a more complex representation of the laws of thought that the one provided by the usual Boolean theory of logical connectives. This representation is the *Intelligence Code*. In this conception the meaning of *intelligence* is essentially related to self-reference, i.e. related to recognition. [21] The Intelligence Code is related to quantum mechanics for two-state sytems and to quantum fields. Matrix logic is naturally quantized, since its eigenvalues take discrete values which are $\pm 1, 0, 2, \pm\Phi$, with Φ the Golden number [26]. In this setting, the null quantum-cognitive observable is the 2×2 matrix, $\mathbf{0}$, with identical entries given by 0, the *mind apeiron*. The relation with quantum field operators and this observable which represents the apeiron observable, is their role in polarizing this cognitive-quantum apeiron

[21] There is no cognition unvinculated to a subject, in contrast with the basic tenants of the Theory of Information. In other words, cognition is embodied, instead of received. The latter conception is, of course, another example of the Cartesian Cut, which proposes a receiver, a desingularized unstructured subject, instead of a cocreator of meaning. In that alienated conception, there is no actual physics, and the bottomline is the replacement of measurements and recognition by registrations. These are carried out by a subject turned into an object, operating as a physics independent machine. This is also the (mis)conception operating in the current standard dogma of genetics, which stands in sharp contrast with wave genetics [37].

through non-null square roots which can be represented by plenumpotents, i.e. Musès hypernumbers whose square is $\mathbf{0}$. In distinction with the other cognitive-quantum observables, is that the eigenstates of $\mathbf{0}$ are no longer quantized, but rather give an orthogonal complex two-dimensional nullvector space. In this way the plenum is no longer represented by a single point, $\mathbf{0}$, but rather becomes an extended object or zero-brane. This will allow to map the twistor representations of the extended photon presented in (120) with its representation in a cognitive state and viceversa!

6.2 Torsion of Cognitive Space, Schroedinger Entanglement, Non-Orientable Manifolds, the Klein Bottle, Quantum Field Theory, Logic and Hypernumbers

We consider a space of all possible cognitive states (which in this context replace the Boolean logical variables) represented in this plenum as the set of all Dirac bras $< q| = (\bar{q} \quad q)$, and kets $|q>$, with $\bar{q} + q = 1$, $q \in \mathbf{R}$ [22], is a continuous cognitive logical value not restricted to the false and true scalar values, represented by the numbers 0 and 1 respectively. Still, the standard logical connectives admit a 2×2 matrix representation of the their 'truth tables' and now we have that for such an operator, L, we have the action of L on a ket $|q>= \begin{pmatrix} \bar{q} \\ q \end{pmatrix}$ is denoted by $L|q>$ alike the formalism in quantum mechanics, and still we have a scalar truth value given by $< p|L|q>$, where $< p|$ denotes another logical vector. We can further extend the usual logical calculus by considering the Truth and False operators, defined by the eigenvalue equations $\mathrm{TRUE}|q>= |1>$

[22] Notice that a difference with the definition of qubits in quantum computation, is that for them we have the normalization condition for complex numbers of quantum mechanics. In this case, the values of q are arbitrary real numbers, which leads to the concept of non-convex probabilities. While this may sound absurd in the usual frequentist interpretation, when observing probabilities in non-orientable surfaces, say, Moebius surfaces, it turns out to be very natural. If we start by associating to both sides of an orientable surface -from which we construct the Moebius surface by the usual procedure of twisting and gluing with both sides identified- the notion of say Schroedinger's cat being dead or alive in each side, then for each surface the probability of being in either state equals to 1 and on passing to the non-orientable case, the sum of these probabilities is 2. While this is meaningless in an orientable topology, in the non-orientable case which actually exist in the macroscopic world, this value is a consequence of the topology. In this case, the superposed state ' being alive and being dead' or 'true plus false' which is excluded in Aristotelian dualism by the principle of non-contradiction, is here the case very naturally supported by the fact that we have a non-trivial topology and non-orientability. As for the case of negative probabilities, we see in the previous example that -1 is the probability value complement of the value 2.

and FALSE$|q>=|0>$, where $|1>=\begin{pmatrix}0\\1\end{pmatrix}$ and $|0>=\begin{pmatrix}1\\0\end{pmatrix}$ are the true and false vectors. It is easy to verify that the eigenvalues of these operators are the scalar truth values of Boolean logic. We can represent these operators by the matrices

$$\mathrm{TRUE}=\begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}, \mathrm{FALSE}=\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} \tag{126}$$

We note that the spaces of bras and kets do not satisfy the additivity property of vector spaces -while keeping the property that one is the dual of the other- due to the fact that normalization is not preserved under addition. A superposition principle is necessary. If $|p<$ and $|r>$ are two normalized states, then the superposition defined as follows

$$|q>=c|p>+\bar{c}|r>, \quad \text{where} \quad \bar{c}+c=1, \tag{127}$$

also defines a normalized logical state. We can interpret these coefficients as components of a logical state $|c>$ or still a probability vector, termed *denktor*, a German-English hybrid for a *thinking vector*. The normalization condition is found as follows: Multiply the states $|p>$ and $|r>$ by $\bar{c}$ and c, respectively. By definition, the normalization condition on the sum $|q>$ with coefficients $\bar{c}, c$ leads to

$$\begin{pmatrix}\bar{q}\\q\end{pmatrix}=c\begin{pmatrix}\bar{p}\\p\end{pmatrix}+\bar{c}\begin{pmatrix}\bar{r}\\r\end{pmatrix}=\begin{pmatrix}c\bar{p}+\bar{c}\bar{r}\\cp+\bar{c}r\end{pmatrix}, \tag{128}$$

yet, since $\bar{q}+q=c\bar{p}+\bar{c}\bar{r}+cp+\bar{c}r=c(\bar{p}+p)+\bar{c}(\bar{r}+r)=c.1+\bar{c}.1$ and thus $c+\bar{c}=1$ since $|q>$ is a normalized state by assumption. So through this superposition principle is that we can give a vector space structure to normalized cognitive states. We now can identify under these prescriptions, the tangent space to the space of bras (alternatively, kets) with the space itself. [23]

Returning to the vector space structure provided by the superposition principle, and thus the identification of its tangent space with the vector space itself, it follows that a vector field as a section of the tangent space can be seen as a transforming a bra (ket) vector into a bra (ket) vector through a 2×2 matrix, so we can identify the tangent space which with the space of logical operators. We have as usual the commutator of any such matrices $[A,B]=AB-BA$ and

[23] Here it is simple to see that if $|q>, |q'>$ are two superpositions, then for any operator L, $L|(q+q')>=L|q>+L|q'>$.

the anticommutator $\{A, B\} = AB + BA$. In particular we take the case of $A = \text{FALSE}, B = \text{TRUE}$ and we compute to obtain

$$\begin{aligned} [\text{FALSE}, \text{TRUE}] &= \text{FALSE} - \text{TRUE}, & (129)\\ \{\text{FALSE}, \text{TRUE}\} &= \text{FALSE} + \text{TRUE}. & (130) \end{aligned}$$

Thus in the subspace spanned by TRUE and FALSE we find that the commutator that here coincides with the Lie-bracket of vectorfields defines a torsion vector given by the vector $(1 \ -1)$, and that this subspace is integrable in the sense of Frobenius: Indeed, $[\text{FALSE}, \text{TRUE}], \text{TRUE}] = [\text{FALSE}, \text{TRUE}]$ and $[[\text{FALSE}, \text{TRUE}], \text{FALSE}] = [\text{TRUE}, \text{FALSE}]$. Furthermore, on account that $\text{TRUE}^2 = \text{TRUE}$ and $\text{FALSE}^2 = \text{FALSE}$, i.e. both operators are idempotent, then the anticommutators also leaves this subspace invariant.

The remarkable aspect here is that the quantum distinction produced by the commutator, exactly coincides with the classical distinction produced by the difference (eq. (25)), while the same is valid for the anticommutator with a classical distinction which is represented by addition (eq, (26)). We notice that in distinction of quantum observables, these logical operators are not hermitean and furthermore they are noninvertible. Furthermore, we shall see below how this torsion is linked with the creation of cognitive superposed states, very much like the coherent superposed states that appear in quantum mechanics. Now, if we denote by M the commutator $[\text{FALSE}, \text{TRUE}]$ so that from eqs. $(22, 25)$ we get

$$M = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \tag{131}$$

we note that it is nilpotent, (in fact a nilpotent hypernumber, since $M = \epsilon_2 + i_1 = \sigma_z + i_1$)

$$M^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \equiv \mathbf{0}, \tag{132}$$

thus yielding the identically zero matrix. $\mathbf{0}$ represents the universe of all possible cognitive states created by a non-null divisor of zero; we have already called it the mind apeiron. M creates a polarization of the mind apeiron through the fact that the torsion is a superposed state which cannot be fit into the scheme of Boolean logic but can be obtained independently by the loss of orientability of a surface which thus allows for paradox. Since M coincides with the classical difference between FALSE and TRUE, which are not hermitean, then we can think of this non-invertible operator as a cognitive operator related to the variation of truth value of the cognitive state, as we shall prove further below that $M = -\frac{d}{dq}$.

We would like to note that this polarization of the plenum $\mathbf{0}$ is not unique, there are many divisors of $\mathbf{0}$, the mind apeiron, for instance the operator

$$\text{ON} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} := a^{\dagger}, \tag{133}$$

and

$$\text{OFF} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} := a \tag{134}$$

satisfy

$$a^2 = \mathbf{0}, (a^{\dagger})^2 = \mathbf{0}, \tag{135}$$

and furthermore, $\{a, a^{\dagger}\} = I$, so they can be considered to be matrix representations of creation and annihilation operators, $a^{\dagger}$ and a as in quantum field theory. In fact, if we consider the wave operators given by the exponentials of $a, a^{\dagger}$ we have

$$e^a = I + a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \text{IMPLY}, e^{a^{\dagger}} = I + a^{\dagger} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \text{IF}, \tag{136}$$

where IMPLY $=\rightarrow$ is the implication, and IF $=\leftarrow$ is the converse implication: $x \leftarrow y = \bar{x} \rightarrow \bar{y}$. Thus the implication and the converse implication logical operators are both wave-like logical operators given by the exponentials of divisors of $\mathbf{0}$, and in fact they are derived from quantum field operators of creation and annhilation in second-quantization theory, $a^{\dagger}$ and a, respectively, which in fact can be represented by nilpotent hypernumbers. Indeed, $a = \frac{1}{2}(\epsilon_3 - i_1) = \frac{1}{2}(\sigma_x - i_1)$ and $a^{\dagger} = \frac{1}{2}(\epsilon_3 + i_1) = \frac{1}{2}(\sigma_x + i_1)$; see [13].

6.3 The Quantization of Matrix Logic

Now we wish to prove that the interpretation of M as the logical momentum operator is natural since $M = -\frac{d}{dq}$. Indeed,

$$-\frac{d}{dq}|q >= -\frac{d}{dq}\begin{pmatrix} 1-q \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} \bar{q} \\ q \end{pmatrix} = M|q > \tag{137}$$

so that for any normalized cognitive state $|q>$ we have the identity

$$M = -\frac{d}{dq}, \tag{138}$$

which allows to interpret the cognitive operator as a kind of logical momentum. Thus, in this setting which is more general but less primitive than the calculus of distinctions from which it can be derived [13], it is the non-duality of TRUE and FALSE what produces cognition as variation of the continuous cognitive state; cf. footnote no. 1. We certainly are here with a situation that is far from the one contemplated by Aristotle with his conception of a trivial duality of (scalar) true and false, and which lead the elimination (and consequent trivialization) of time and of subjectivity, as argued in [13].

Now consider a surface given by a closed oriented band projecting on the xy plane. Thus to each side of the surface we can associate its normal unit vectors, $(1 \;\; 0)$ and $(0 \;\; 1)$. Suppose that we now cut this surface and introduce a twist on the band and we glue it to get thus a Moebius surface. Now the surface has lost its orientability and we can identify one side with the other so that we can generate the superpositions

$$< 0| + < 1| =< (1 \;\; 1)| =< S_+|, \quad < 0| - < 1| =< (1 \;\; -1)| =< S_-|. \tag{139}$$

which we note that the latter corresponds to the torsion produced by the commutator of TRUE and FALSE operators. Theses states are related by a change of phase by rotation of 90 degrees. What the twisting and loss of orientability produced, can be equivalently produced by the fact that TRUE and FALSE are no longer dual as in Boolean logic and the Aristotelian frame. What is relevant is their difference (and we return to the Introduction's motto of a difference that produces differences), which in the case of scalar truth values does not exist. The other state also can be interpreted as a state that represents the fact that the states as represented by vectors, have components standing for truth and falsity values which are independent, so that the Aristotelian link that makes one the trivial reflexive value of the other one is no longer present: they each have a value of their own. In that case then $(0 \;\; 0)$ is another state, 'false and true' (which is the case of the Liar paradox as well as Schroedinger's cat), which together with $(1 \;\; 1)$, 'nor false nor true' state together with $(0 \;\; 1)$, true, and $(1 \;\; 0)$ false states we have a 4-state logic in which the logical connectives have been promoted to operators.

Now consider for an arbitrary normalized cognitive state q the expression

$$\begin{aligned} [q, M]|q> &= [q, -\frac{d}{dq}]|q >= -q\frac{d}{dq}|q > +\frac{d}{dq}q|q >= -q\frac{d}{dq}\begin{pmatrix} 1-q \\ q \end{pmatrix} \\ &+ \frac{d}{dq}\begin{pmatrix} q-q^2 \\ q^2 \end{pmatrix} = \begin{pmatrix} q \\ -q \end{pmatrix} + \begin{pmatrix} 1-2q \\ 2q \end{pmatrix} = |q >, \end{aligned} \tag{140}$$

for any normalized cognitive state q so that we have the quantization rule

$$[q, M] = I, \tag{141}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity operator. Instead of the commutation relations of quantum mechanics $[q, p] = i\hbar$ for $p = -i\frac{\partial}{\partial q}$ and those of diffusion processes associated to the Schroedinger equation, $[q, p] = \sigma$ where $p = \sigma\frac{\partial}{\partial x}$ with σ the diffusion tensor given by the square-root of the metric g on the manifold with coordinates x on which the diffusion takes place so that $\sigma \times \sigma^\dagger = g$ [7], we have that the commutation of a normalized cognitive state with the cognitive (momentum) operator is always the identity yielding thus a fixed point. Indeed, consider the function $F_M(q) = [q, M]$, then $F_M(F_M(F_M(FM(\ldots))))(|q>) = |q>$, for any normalized cognitive state $|q>$. Thus, $F_M(q)$ defines what is called in system's theory an eigenform, albeit one which does not require infinite recursion but achieves a fix point already in the first step of the process, by the formation of the commutator $[q, M]$ [67]. This is the structure of the Self, which whatever operation may suffer by the action of logical operators, it retains its invariance by the quantization of logic as expressed above by eq. (141).

Now we want to return to the superposed states, S_+ and S_-, the latter being the torsion produced by the commutator of the TRUE and FALSE operators, to see how they actually construct the cognitive operator. First a slight detour to introduce the usual tensor products of two cognitive states, $|p><q|$ which as the tensor product of a vector space and its dual is isomorphic to the space of linear transformations between them, we can think as an operator L acting by left multiplication on kets and by right multiplication on bras. So that if $L = |p><q|$ then $<y|L|x> = <y, p><q|x>$, for any $<y| = \bar{y} < 0| + y < 1|$ and $|x> = \bar{x}|0> + x|1>$, where $<x|y> = \delta_{xy}$ equal to 1 for $x = y$, and equal to 0 for $x \neq y$ and $\sum_i |x_i><x_i| = I$. Then,

$$M = |S_+><S_-|, \tag{142}$$

which shows that the cognitive operator that arises from the quantum-classical difference between the TRUE and FALSE operators can be expressed in terms of the tensor products of the superposition states, being the sum of the true and false states and the torsion produced in the quantum commutator of the TRUE and FALSE operators.

Starting with the logical momentum M, that satisfies $[q, M] = I$ for any cognitive variable q, we can link the quantization rule in cognitive space to the quantization rule of Bohr-Sommerfeld. The logical potential carrying the logical energy could be linked to the Bohr energy of atomic structures in the following way: $\infty(k) = \oint M dq = 2\pi(n + 1/2) = k\pi$, where q is a logical variable (if it is zero than the contour integral runs a full great circle on the Riemann sphere of zeros), n is the winding number specifying the numbers of times the closed curve runs round in an anticlockwise sense; n runs the bosonic numbers $0, 1, 2 \ldots$ and $(n + 1/2)$ the fermionic numbers, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. The topological potential is an odd multiple $(2n + 1)\pi$ of the elemental (topo)logical phase π and is $\hbar^{-1}$ times the Bohr energy of the quantum oscillator: $\oint p dx = 2\pi\hbar(n + 1/2)$, where the position and momentum operator satisfy the standard quantum commutation relation: $[x, p] = i\hbar$. As we see, the topological potential, multiplied by the factor $\hbar$, gives the Bohr quantum energy opening up the possibility to treat atomic structure as a dynamical logic in a fundamental sense, where quantization stems from the closed topology or self-observation feature at this fundamental level of reality. Another interesting conjecture which follows is, since matter, as energy, ($E = mc^2$) is a topologically transformed logical energy, the mass of an object is basically the information contained in the holomatrix which projects it out from the ground state.

6.4 The Time and Spin Operators, Quantum Mechanics and Cognition

Let us now introduce the operator defined by

$$\triangle = a - a^\dagger \tag{143}$$

so that is follows that its matrix representation is

$$\triangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{144}$$

and furthermore

$$\triangle = \rightarrow - \leftarrow . \tag{145}$$

We shall call $\triangle$ the TIME operator. [24] We notice that it is unitary and antisymmetric:

$$\mathrm{TIME}^\dagger = \mathrm{TIME}^{-1} = -\mathrm{TIME}. \tag{146}$$

[24] Remarkably, $-2i$TIME is the hamiltonian operator of the damped quantum oscillator in the quantum theory of open systems; see N. Gisin and I. Percival, arXiv:quant-ph/9701024v1. In this theory based on the stochastic Schroedinger equation the role of torsion is central [10] (2007)].

As an hypernumber TIME $= -i_1$, minus the unique 2×2 matrix representing a 90 degrees rotation, the old commutative square root of -1 from which complex numbers appeared. The reason for considering this operator given by the difference of nilpotents is because it plays the role of a comparison operator. Indeed, we have

$$< p|\mathrm{TIME}|q > \quad = \quad \bar{p}q - \bar{q}p = (1-p)q - (1-q)p = q - p = \bar{p} - \bar{q}. \quad (147)$$

TIME appears to be unchanged for unaltered states of consciousness:

$$< q|\mathrm{TIME}|q >= 0, \quad (148)$$

and if we have different cognitive states p, q, then $< p|\mathrm{TIME}|q > \neq 0$. So this operator does represent the appearance of a primitive difference on cognitive states (another example of the motto in the Introduction). It is antisymmetric and unitary. It is furthermore linked with a difference between annihilation and creation operators and thus stands for what we argued already as a most basic difference that leads to cognition and perception: the *appearance of quantum jumps*. Without them, no inhomogeneities nor events are accesible to consciousness. The very nature of self-reference as consciousness of consciousness requires such an operator for the joint constitution of the subject and the world. Thus its name, a TIME operator operator. It stands clearly in the subject side of the construction of a conception that overcomes the Cartesian cut, yet a subject that has superposed paradoxical states. Yet, we have seen that it plays a major role in the representation of the extended photon.

Let us consider next the eigenvalues of TIME, i.e. the numbers λ such that $\mathrm{TIME}|q >= \lambda|q >$; they are obtained by solving the characteristic equation $\det|\mathrm{TIME} - \lambda I| = \lambda^2 + 1 = 0$, so that they are $\lambda = \pm i$ with complex eigenstates

$$\begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}. \quad (149)$$

They are not orthogonal, but self-orthogonal; thus, they are spinors, and the complex space generated by them generates a two-dimensional null space. We diagonalize TIME by taking

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \mathrm{TIME} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (150)$$

so that

$$\mathrm{TIME}_{\mathrm{diag}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (151)$$

which as an hypernumbers we have that $\text{TIME}_{\text{diag}} = i_2$, so that $\text{TIME}^2_{\text{diag}} = -I$. We want finally to comment that TIME is not a traditional clock,yet it allows to distinguish between after and before ($\rightarrow - \leftarrow$), forward and backwards. There is no absolute logical time, nor a priviliged direction of it. To have a particular direction it must be asymetrically balanced towards creation or annihilation. This can be computed as the complement of the operator phase[25]

$$\overline{cos(2a^\dagger) + sen(2a^\dagger)} = a^\dagger - a, \tag{152}$$

from which it follows that $\text{TIME} = \overline{\leftarrow^2} = \rightarrow - \leftarrow$, as we stated before.

Let us now retake M and decompose it as

$$\begin{aligned} M &= \text{TIME} + \sigma, \quad \text{or still} && (153) \\ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. && (154) \end{aligned}$$

Then we have that

$$< q|M|q > = < q|\sigma|q > . \tag{155}$$

Indeed, since $< q|\text{TIME}|q >= 0$, so that the proof of eq. (155) follows. Furthermore we note that

$$< q|\sigma|q >= \bar{q}^2 - q^2 = (\bar{q} - q)(\bar{q} + q) = \bar{q} - q. \tag{156}$$

from the normalization condition. Note here that the identity given by eq. (156) is a kind of quadratic metric in cognitive space which due to the normalization condition looses its quadratic character to become the difference in the cognitive values: $\bar{q} - q = 1 - 2q$ which becomes trivial in the undecided state in which $\bar{q} = q = \frac{1}{2}$.

The role of σ is that of a SPIN operator, as we shall name it henceforth, which coincides with the hypernumber ϵ_2 (or as a Pauli matrix is σ_z), so that $\sigma^2 = I$ the non-trivial square root of hypernumber $I = \epsilon_0$, which is the usual Pauli matrix σ_z in the decomposition of a Pauli spinor in the form $\sigma_x e_x + \sigma_y e_y + \sigma_z e_z$, for e_x, e_y, e_z the standard unit vectors in $\mathbf{R}^3$ and we write their representations as hypernumbers

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \epsilon_3, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \epsilon_1, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \epsilon_2. \tag{157}$$

[25]The complement of a logical operator L, is defined by $\bar{L} = I - L$.

We can rewrite this average equation $< q|M|q >=< q|\sigma|q >$ as an average equation which the l.h.s. takes place in cognitive space of normalized states $|q >$ and the r.h.s. in a Hilbert space of a two-state quantum system, say, spin-up $\psi(\uparrow)$, spin-down $\psi(\downarrow)$, so that the generic element is of the form

$$\psi = \psi(\uparrow)|0 > +\psi(\downarrow)|1 > . \tag{158}$$

Indeed, if we write

$$|q >= \overline{\psi(\uparrow)}\psi(\uparrow)|0 > +\overline{\psi(\downarrow)}\psi(\downarrow)|1 >, \tag{159}$$

then the r.h.s. of eq. (156) is $\bar{q}^2 - q^2$, with $\bar{q} = \overline{\psi(\uparrow)}\psi(\downarrow)$, and $q = \overline{\psi(\downarrow)}\psi(\uparrow)$, so that eq. (156) can be written as

$$< q|M|q >=< \psi|\sigma|\psi > \tag{160}$$

where the average of M is taken in cognitive states while that of the SPIN operator is taken in the two-state Hilbert space.

We review the previous derivation for which the clue is the relation between cognitive states $|q >$ and elements of two-state of Hilbert state $|\psi >$: The former are derived from the latter by taking the complex square root of the latter. Hence, probability$(|0 >) = \bar{q} = \overline{\psi(\uparrow)}\psi(\uparrow)$ and probability$(|1 >) = q = \overline{\psi(\downarrow)}\psi(\downarrow)$, so that $< \psi|\sigma|\psi >= \bar{q} - q = (\bar{q} - q)(\bar{q} + q) = \bar{q}^2 - q^2$. Therefore, by using the transformation between real cognitive states q defined by the complex square root of ψ, i.e. $q = \bar{\psi}\psi$, we have a transformation of the average of the cognitive operator M on cognitive states on the average of SPIN on two-states quantum elements in Hilbert state, i.e. eq. (159). This is a very important relation, established by an average of the cognition operator (which transforms an orientable plane into a non-orientable Moebius surface due to the torsion introduced by M, as represented by eq.(139), and SPIN on the Hilbert space of two-state quantum mechanics. It is an identity between the action of the cognizing self-referential mind and the quantum action of spin. Thus the cognitive logical processes of the subject become related with the physical field of spin on the quantum states. This is in sharp contrast with the Cartesian cut, and we remark again that this is due to the self-referential classical-quantum character of M as evidenced by eq. (139) which produces a torsion on the orientable cognitive plane of coordinates (true, false) to one to yield a superposed state, S_{-}. The relation given by eq. $(155, 159)$ establishes a link between the operations of cognition and the quantum mechanical spin. This link is an interface between the in-formational and quantum realms, in which topology, torsion, logic and the quantum world operate jointly. Yet, due to fact that for the Klein bottle there is no inside nor outside, the exchange can go in both ways, i.e. the quantum realm can be incorporated into the classical cognitive dynamics, while the logical elements can take part in the quantum evolution. Indeed, if we have a matrix-logical string which contains the momentum product, say, $\ldots < x|A|y >< q|M|q >< z|B|s > \ldots = \ldots < x|A|y >< \psi|\sigma|\psi >< z|B|s > \ldots$. Thus, the factor $< \psi|\sigma|\psi >$ entangles with the rest of the classical logical string creating a Schrodinger cat superposed state, since we have a string of valid propositions where one may be the negation of the other [26]

6.5 The Klein Bottle, Quantum Computation and the Intelligence Code

There is still another very remarkable role of these superposed states in producing a topological representation of a higher order form of self-reference, produced from oppositely twisted Moebius surfaces. So we shall consider the Cartesian modulo 2 sum of the superposed states

$$\mathcal{H} := |S_+ > \oplus |S_- >= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{161}$$

which we call the topological in-formation operator which is an hypernumber; indeed, $\mathcal{H} = \sigma_x + \sigma_z = \epsilon_3 + \epsilon_2$. We could have chosen the opposite direct sum or still place the minus sign on the first row in any of the columns and obtain a similar theory, but for non-hermitean operators unless the minus sign is on the first matrix element. Notice that it is a hermitean operator, which essentially represents the topological (or still, logo-topological) in-formation of a Klein bottle formed by two oppositely twisted Moebius surfaces [72]. [27] The information matrix satisfies $\mathcal{H}\mathcal{H}^\dagger = \mathcal{H}\mathcal{H}^{-1} = 2I$. We recognize in taking $1\sqrt{2}\mathcal{H}$ the Hadamard gate in quantum computation [36], which due to the introduction of the $\frac{1}{\sqrt{2}}$ factor is hermitean and unitary. Now we have two orthogonal basis given

[26] This primordial role of spin as as protopsychic as well as protophysical is found also in [27], though not mathematically based. In this work it is claimed that spin is “the linchpin between mind and brain”, though in a certain Cartesian way, associating spinor fields to processes in the brain and not to the processes of the mind. They further link it with self-referential processes alike the Klein bottle [27]; see also [82].

[27] Alternatively we can introduce instead of $\mathcal{H}$ another in-formation matrix for the Klein bottle, namely

$$.\mathcal{H} := |S_+ > \oplus |S_- >= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{162}$$

which is non-hermitean.

by the sets $\{|0>, |1>\}$ and $\{|S_->, |S_+>\}$ of classical and superposed states respectively, the latter un-normalized for which a factor $\frac{1}{2}$ has to be introduced but still does not give a unitary system as in quantum theory. An important role of the Klein bottle is precisely to transform these orthogonal basis, from classical states to superposed states which are nor classical nor quantum, but become quantized by appropiate normalization with the $\frac{1}{\sqrt{2}}$ factor. Indeed,

$$\mathcal{H}|0>=|S_+>, \mathcal{H}|1>=|S_->, \tag{163}$$

and

$$\frac{1}{2}\mathcal{H}|S_+>=|0>, \frac{1}{2}\mathcal{H}|S_->=|1> . \tag{164}$$

In the logical space coordinates $(true, false)$ we have rotated the state $|0>$ clockwise by 45 degrees through the action of $\mathcal{H}$ and multiplied it its norm by 2, and for the state $|1>$ we have rotated it likewise after being flipped. In reverse, the superposed states are transformed into the classical states by halving the in-formation matrix of the Klein bottle, producing 45 degrees counterclockwise rotations, one with a flip. Now classical and quantum states are functionally complete sets of eigenstates spanning each other. The classical states $|0>$ and $|1>$ can be easily determined to be the eigenstates of AND, and and the superposed states $|S_->$, $|S_+>$ are the eigenstates of NOT. It is known that the logical basis of operators $\{\mathrm{AND}, \mathrm{OR}\}$ is functionally complete, generating all operators. Hence our system of classical and superposed (or still, quantum by appropiate normalization by $\frac{1}{\sqrt{2}}$) eigenstates constitute together a functionally complete system: all operators of matrix logic can be obtained from them. This system is self-referential. Furthermore, there are operators which produce the rotation of one orthogonal system on the other orthogonal system. The logical differentiation operator M defined by the commutator $[\mathrm{FALSE}, \mathrm{TRUE}]$ or still eq. (129) transforms classical states $|x>=\bar{x}|0>+x|1>$ into $|S_-$ and still the anticommutator $\{\mathrm{FALSE}, \mathrm{TRUE}\}$ which coincides with the matrix $\mathbf{1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ transforms $|x>$ into $|S_+>$, i.e.

$$M|x>=|S_->, \mathbf{1}|x>=|S_+> . \tag{165}$$

which can be rephrased by saying that M evidences on its action on a classical state the torsion in the quantum commutator of FALSE and TRUE while the ONE operator $\mathbf{1}$ transforms $|1>$ into $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |S_->$. Since both M and $\mathbf{1}$ are non-invertible, we shall use instead the fact that $\mathcal{H}^{-1} = \frac{1}{2}\mathcal{H}$, so that in addition of the transformation by the Klein bottle of the classical basis in eq. (163), the reversed transformation from the superposed to the classical states is achieved by

$$\frac{1}{2}\mathcal{H}|S_+ >= |0>, \frac{1}{2}\mathcal{H}|S_- >= |1> . \tag{166}$$

Yet, we stress again that these transformations are not unitary which is easily resolved by the $\frac{1}{\sqrt{2}}$ factor and then we have a transformation of classical into quantum states and viceversa. In the latter case, the renormalized Klein bottle acts like a quantum operator producing coherent quantum states, a topological Schroedinger "cat" state which does not decohere. Therefore to resume, from these four states, or alternatively, from the matrix representation of the Klein bottle, it is possible to generate the Intelligence Code [26, 13]. We note that it is essentially self-referential.

7 The Eigenstates of the Cognitive Plenum, Twistors and the Extended Photon, and the So-Called Mind-Matter Problem

We shall now discuss the eigenstates of the mind apeiron, namely the 2×2 identicaly zero matrix which we denoted as $\mathbf{0}$; this was first partially and roughly sketched in [25]. In distinction with the other logical operators the eigenstates of $\mathbf{0}$, as a linear transformation from $\mathbf{C}$ on $\mathbf{C}$, which thus becomes a point of $\mathbf{C}^2$, its origin, are no longer quantized, but rather give an orthogonal complex two-dimensional nullvector space. [28] In this way the plenum is no longer represented by a single point, $\mathbf{0}$, but rather becomes an extended object or zero-brane. This phenomenon of the blowup of a point or more generally a manifold (here $\mathbf{C}^2$) is well known in complex Clifford bundles and is the most fundamental operation in algebraic geometry (which was the origin of twistors by R. Penrose [22]). Namely, it consists in replacing each point of the manifold by the projectivized tangent space at that point [68]. In the case of the mind plenum represented by the origin in $\mathbf{C}^2$ it amounts to replace it by the projectivized tangent space at the origin. Let Z be the origin in n-dimensional complex space, $\mathbf{C}^n$. That is, Z is the point where the n coordinate functions $(x_1, \ldots, x_n) \in \mathbf{C}^n$ simultaneously vanish. Let

[28] We have already seen this an identical situation in the eigenstates of TIME. Thus the eigenstates of the mind apeiron are given by a plenumpotence condition alike the eikonal equation; we shall see that this similarity is in fact an identity.

$\mathbf{CP}(n-1)$ be the (n-1)-dimensional complex projective space with homogeneous coordinates $(y_1, \ldots, y_n)$. The blowup of Z is the subset of $\mathbf{C}^n \times \mathbf{CP}(n-1)$ that satisfies the equation $x_i y_j = x_j y_i$, for all $i, j = 1, \ldots, n$. In the case $n = 2$ where (y_1, y_2) are complex numbers not both zero, homogeneous coordinates of $\mathbf{CP}(1)$, which can thus be also can described by the single coordinate $\xi = \frac{y_1}{y_2}$. Since $\mathbf{CP}(1)$ is the familiar Riemann (-Argand-Euler) sphere of complex analysis, S, then the blowup of the origin in $\mathbf{C}^2$ is its replacement by the Riemann sphere, or still by the complex 2-sphere, S^2, on which we represent the spinor eigenstates of the mind apeiron.[29] Indeed, a cross-section of the blowup of the origin in $\mathbf{C}^2$ represents the spinor vectors in S or in its isomorphic two-sphere, S^2, giving a 2- complex-dimensional vector space, which can be mapped to the 2-dimensional logic space of matrix logic by stereographic projection. [30]. We apply this to the twistor representation of the extended photon through the maximal monochromatic algebra as described by (118, 119) which has an equivalent representation as a pair of divergenceless orthogonal spinor vectors $(\omega^A, \pi^{A'})$, $A, A' = 1, 2$ by (120). By stereographic projection of the spinors ω^A, π^A which form the twistor representation of the extended photon, we obtain a representation of it in cognitive space in the basis $|0>$ and $|1>$. Viceversa, by taking the inverse of the stereographic

[29] There is a certain ambiguity on regards of $\mathbf{0}$ being also interpretable as the origin in $\mathbf{C}^4$ rather $\mathbf{C}^2$, after all it has four entries! In this case, the blowup of the origin in $\mathbf{C}^4$ has no longer for crossection $\mathbf{CP}(1)$ but instead $\mathbf{CP}(3)$, which is the three dimensional complex projective space of twistors [22] (1979). In this case, the eigenstates of $\mathbf{0}$ are (projective) twistors, elements of a nullspace. The difference in this interpretation is that for the effect of the association between the maximal monochromatic algebra of the extended photon is characterized by eqs. (120) representing the *pair* of spinors characterized by eqs. (118, 119) as cross-sections of the blowup of $\mathbf{0}$ as the origin in $\mathbf{C}^2$.

[30] The blowing up of the origin, transforming its point-like structure to yield a manifold has profound consequences. For example, the blowing up of the origin in $\mathbf{R}^2$ is the Moebius surface, which as we already saw is basic to the generation of Intelligence Code: its normal vectors defines logical momentum and also the Klein bottle. We recall that two oppositely twisted Moebius bands generate the Klein bottle [72], the high order (in relation to the Moebius band [28]) surface of paradox, whose matrix representation, up to a normalizing constant, is the Hadamard gate of quantum computation, which together with the phase conjugator, allows to generate all quantum gates [36]. Thus, embedded in the blowup of the mind apeiron as the origin of $\mathbf{C}^2$ lies the generation of the Intelligence Code from the blowup of the origin in $\mathbf{R}^2$. DNA performs quantum computations [37] which is further related to holography [38, 39]. We recall that holography is already performed by the Klein bottle visual processing of the neurocortex. This evidences the importance of the Hadamard gate in quantum computation and the Klein bottle multivalued logic we presented. It can also be derived from the semiotic codification of torsion as a distinction and the time-waves related to paradox; see [13]. For technological implementations of the Moebius band, the Klein bottle and the generation of Kozyrev fields see [85]; for its relation to anthropology [79]. An important contribution to the geometrical studies of consciousness, though in a different setting is [86].

projection we reconstruct the maximal monochromatic algebra. In any case via the normalized Klein bottle Hadamard in-formation matrix, all the operators of matrix logic are generated (the completitude we mentioned before). In this we see how the extended photon which we claimed to be a subjective-objective fused structure-process is represented as a basis for cognitive space, and conversely, from cognitive space we are able to codify the maximal monochromatic algebra representation of the extended photon. This establishes the full self-referential construction of a world which is perceived through quantum jumps, i.e. distinctions, or still, in terms of cognitive states that belong to states of cognition of the mind. Yet, we have seen above that the role of the Planck constant $\hbar$ is precisely to connect the transformation of the quantum world into the world of the mind, bridging thus the material and mind domains. Since $\hbar$ can be associated to a cosmological scale [30, 31], we can speak about *cosmological consciousness.* Thus, we can modify the quotation in the Introduction, "...light is seeing", to light is seeing-thinking, as these two actions become inseparable at the mind apeiron level. More complex levels operate through convolution, and perhaps through other processes. The Riemann sphere is not only instrumental to this joint constitution by codifying the extended photon as a cognitive state. It is also the manifold in which the logarithmic function takes multivalued complex values to quantize the quantum jumps in terms of the different branches of the logarithm, allowing thus to codify the 'outer' and 'inner' worlds. For further ellaborations in relation to the transactional interpretation of Quantum Mechanics, the important notion of anticipatory systems [80], cosmological Kozyrev torsion fields and entanglement, and brain synchronization in binocular vision, see [43]. [31]

For closing remarks is enough to summarize by saying that the plenumpotence of 0 and **0** have been shown to be related to quantum jumps through the *extended* singularities of the torsion field related to propagating *extended* eikonal singularities and to the *extended* eigenstates of the mind apeiron, enclosing and generating a self-referential world which amounts to the *extended* character of 0 and that of the subject. This extension is the fusion of the res cogitans and res extensa of Descartes: apeiron.

[31] For a different conception in terms of Endophysics which does not incorporate explicitly self-reference we refer to [40]. The theory of fractal time is relevant [41]. Remarkably, inasmuch matrix logic has a projective structure as well as the eigenstates of the mind apeiron, a theory of altered mind states in terms of Cremona transformations - which arise as well as blowups - has been developed [42].

8 Appendix: Torsion, Non-Commutativity of Space-time and New Energies

We want to introduce torsion in terms of the self-referential definition of the manifold structure in terms of the concept of difference or distinction derived from the operation of comparison to establish a difference that makes a difference, as discussed in the Introduction. We shall assume that there are two observers on a manifold (of dimension n), say observer 1 and observer 2, which may not be moving inertially. To compare measurements and to establish thus a sense of objectivity (identity of their results), they need to compare their measurements which take place in the tangent space at different points of the n-dimensional manifold M in which they are placed, so they have to establish the difference between their reference frames, i.e. the difference between the set of orthogonal (or pseudo-orthogonal) vectors at their locations, the so-called n-beins . Let $e_i(\mathcal{P}_0) = e_i^\alpha(\mathcal{P}_0)\partial_\alpha, i = 1,\ldots,n$ be the basis for observer 1 at point $\mathcal{P}_0$, and similarly $e_i(\mathcal{P}_1) = e_i^\alpha(\mathcal{P}_1)\partial_\alpha$ the reference frame for observer 2 at $\mathcal{P}_1$; let us denote the reference frame at the tangent space to the point $\mathcal{P}_1$ when parallely transported (without changing its length and angle) from $\mathcal{P}_0$ to $\mathcal{P}_1$ by $e_i(\mathcal{P}_0 \to \mathcal{P}_1)$ along a curve joining $\mathcal{P}_0$ to $\mathcal{P}_1$ with an affine connection, whose covariant derivative operator we denote as $\tilde{\nabla}$ as in Section 2. Then, $\tilde{\nabla} e_1$ is the difference between $e_i(\mathcal{P}_0 \to \mathcal{P}_1)$ and $e_i(\mathcal{P}_1)$. This gap defect originates either from: 1) the deformation of $e_i(\mathcal{P}_0)$ along its path to $\mathcal{P}_1$, which cannot be transformed away by a change of coordinates, or, 2) by a change of coordinates from $\mathcal{P}_0$ to $\mathcal{P}_1$, which is not intrinsic and thus can be transformed away, or finally, 3) by a combination of both. Let us move observer's one frame over two different paths. Parallel displacing an incremental vector $dx^b e_b$ from the point $\mathcal{P}_0$ along the basis vector e_a over an infinitesimal distance dx^a to the point $\mathcal{P}_1 = \mathcal{P}_0 + dx^a$ gives the vector

$$e_b dx^b(\mathcal{P}_0 \to \mathcal{P}_1) = dx^b e_b(\mathcal{P}_0) + \Gamma^c_{ba} dx^b \wedge dx^b e_c. \tag{167}$$

Similarly, the parallel transport of the incremental vector $dx^a e_a$ from the point $\mathcal{P}_0$ to $\mathcal{P}_2$ along the frame e_b over an infinitesimal distance dx^b to the point $\mathcal{P}_2 = \mathcal{P}_0 + dx^b$ gives the vector $e_a(\mathcal{P}_1 \to \mathcal{P}_2) = dx^a e_a(\mathcal{P}_1) + \Gamma^c_{ab} dx^a \wedge dx^b e_c$. The gap defect between $e_a(\mathcal{P}_0 \to \mathcal{P}_1)$ and the value of $dx^b e_b(\mathcal{P}_1)$ is

$$dx^b \tilde{\nabla} e_b(\mathcal{P}_1) = dx^b(\frac{\partial e_b}{\partial x^a}) \wedge dx^a - \Gamma^c_{ba} dx^a \wedge dx^b e_c, \tag{168}$$

and the gap defect between the vector $e_b(\mathcal{P}_1 \to \mathcal{P}_2)$ and $e_b dx^b(\mathcal{P}_2)$ is

$$dx^a{}_a(\mathcal{P}_2) = dx^a(\frac{\partial e_a}{\partial x^b}) \wedge dx^a - \Gamma^c_{ab} dx^a \wedge dx^b e_c. \tag{169}$$

Therefore, the total gap defect between the two vectors is (the comparison already mentioned)

$$\begin{aligned} dx^b \tilde{\nabla} e_b(\mathcal{P}_1) - dx^a D e_a(\mathcal{P}_2) &= (\frac{\partial e_b}{\partial x^a} - \frac{\partial e_a}{\partial x^b}) dx^a \wedge dx^b \\ &+ [\Gamma^c_{ab} - \Gamma^c_{ba}] dx^a \wedge dx^b e_c, \end{aligned} \tag{170}$$

where we recognize in the first term the Lie-bracket

$$[e_a, e_b] = (\frac{\partial e_b}{\partial x^a} - \frac{\partial e_a}{\partial x^b}) dx^a \wedge dx^b, \tag{171}$$

which we can write still as

$$[e_a, e_b] = C^c_{ab} e_c, \tag{172}$$

where C^c_{ab} are the coefficients of the anholonomity tensor, and then finally we can write the difference in eq.(170) as

$$dx^b e_b(\mathcal{P}_1) - dx^a \tilde{\nabla} e_a(\mathcal{P}_2) = (C^c_{ab} + [\Gamma^c_{ab} - \Gamma^c_{ba}]) dx^a \wedge dx^b e_c. \tag{173}$$

If we further introduce the vector-valued torsion two form

$$T = \frac{1}{2} T^c_{ab} dx^a \wedge dx^b e_c := \tilde{\nabla} e_b(e_a) - \tilde{\nabla} e_a(e_b) - [e_a, e_b]^c e_c \tag{174}$$

we find that the components T^c_{ab} are given by the so-called torsion tensor

$$T^c_{ab} = C^c_{ab} + [\Gamma^c_{ab} - \Gamma^c_{ba}] \tag{175}$$

Thus, we have two possibilities for the non-closure of infinitesimal parallelograms. Either by anholonomity, or due to the non-symmetricity of the Christoffel coefficients. These are radically different. The former can in some instances be set to be equal to zero, while the other term cannot. Say we have a coordinate transformation continuously differentiable $(x^1, \ldots, x^n) \rightarrow (y^1, \ldots, y^n)$ so we have that an holonomous transformation, i.e. we have that each dy^i is exact of the form

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j. \tag{176}$$

Then, if we take an holonomous basis $e_j = (\frac{\partial y^i}{\partial x^j}) \frac{\partial}{\partial y_i}$, then the anholonomity vanishes, $[e_i, e_j] = 0$ identically on M, and we are left for the expression for the torsion tensor

$$T^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}. \tag{177}$$

Observations. Anholonomity is related to the Sagnac effect and to the Thomas precession [51]. Nowadays, relativistic rotation has become an issue of great interest, and the interest lays in rotating anholonomous frames, in distinction with non-rotating holonomous frames. The torsion tensor evidences how the manifold is folded or dislocated, and the latter situation can be produced by tearing the manifold of by the addition of matter or fields to it. These are the well known Volterra operations of condensed matter physics, initially, introduced in metalurgy [52]. This was the first technological implementation of torsion. The second example was elaborated in the pioneering work by Gabriel Kron in the geometrical representation of electric networks; it lead to the concept of negative resistance [53]. Contemporarily, negative resistance has become an important issue, after the discovery of its existence in some materials, with an accompanying apparent phenomenon of superconductivity [54]. In [10] it was proved that Brownian motions -which are associated to torsion geometries- produce rotational fields. This encompasses the Brownian motions produced by the wave function of arbitrary quantum systems, and the case of viscous fluids, magnetized or not [7, 10, 11]. These examples are independent of any scale, from the galactic to the quantum scales. In the galactic scales, vortices can explain the red-shift without introducing any big-bang hypothesis [66]. Thus, we have a modified form of Le Sage's kinetic theory [65] producing universal fluctuations which have additionally rotational fields associated to them. Due to the universality of quantum wave functions, either obeying the rules of linear or non-linear Quantum Mechanics, Hadronic Mechanics (HM) and Hadronic Chemistry (HQ) [63, 64], these vortices are rather common. Then it is no surprise that vortices and superconductivity appear as universal coherent structures. Superconductivity is usually related to a non-linear Schroedinger equation with a Landau-Ginzburg potential, which is also an example of the Brownian motions related to torsion fields with further noise related to the metric [7, 10]. Furthermore, atoms and molecules have spin-spin interactions which will produce a contribution to the torsion field; we have seen already that the torsion geometry exists in the realm of HM and HQ. [32] This is the case of the compressed hydrogen atom model of the neutron in the Rutherford-Santilli model in HM, in which their is a spin alignment with opposite direction and magnetic moments for the electron and the proton (topologically,

[32] A different approach relates spin-torsion fields [56] to the teleparallel geometries in Minkowski space also explored by the author in [62]. In that work the torsion polarization of the vacuum which also shows up in the phantom DNA effect [37], is related to an hypothetical particle known as the *phyton*. Further experiments related to torsion fields have been carried out [59] and the phenomenae revealed by the Kozyrev experiments have been interpreted in terms of torsion [69, 10].

this yields the Klein bottle), which produces a stable state which leads to fusion [63]. This is *not* a cold fusion process since it appears to occur at temperatures of the order of $5,000$ degrees Celsius. Yet in electrochemical reactions, there are sources of torsion which are given by the the wave functions of the components involved, but furthermore the production of vortex structures. Gas bubbles appear after switching off the electrochemical potentials, and sonoluminescence have been observed at the Oak Ridge National Laboratory, USA [57]. There is a surprising phenomenae of remnant heat that persists after death which could be produced by the vortex dynamics of the tip effect [60]. These experimental findings have been claimed to be observed in different laboratories across the world [55, 60], and explained in terms of torsion fields [58, 59]. Superconductors of class II present also some surprising phenomenae such as low-frequency noise, history-dependent dynamic response, and memory direction, amplitude direction and frequency of the previously applied current [58]. Would these findings be reproduced systematically, we would have a new class of sources of energy, which stem from apeiron. Other important source may occur as a resonant coupling of the torsion generated apeiron Brownian motions with especially designed circuits [83], and in the so-called *cavity structural effect* discovered by research in entomology, which are being developed with widespread applications in Russia and Ukraine [84].

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