Quantizing orthodox gravity

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Abstract: A tentative approach is made to quantizing gravity with a modified renormalization scheme, and is illustrated for the example of a massive scalar field with gravity.

Résumé : Une tentative est faite de quantifier la gravité en utilisant une méthode modifiée de renormalisation. Nous l'illustrons avec un exemple impliquant un champ scalaire en présence d'un champ gravitationnel.

[Traduit par la rédaction]

1. Introduction

With quantum theory and general relativity being such good descriptions of the world, it is somewhat paradoxical that we have still not managed to wed the two theories [1]. Before embarking upon variations, one might question the need to quantize gravity at all, since there is no direct experimental evidence demanding the quantization of the gravitational field.[1] However, if gravity remains classical, since its fields are then not subject to the uncertainty principle of quantum theory, it might be employed to make an indirect measurement of a quantum field that would be more precise than that permitted. This argument for quantizing gravity is not watertight, as one might propose a gravitational coupling to the quantum expectation value, or some other alteration to quantum theory itself [2]. However, it does motivate one to begin by investigating the obstacles to naive quantization of the gravitational field.

The usual scheme of field quantization is plagued by divergences, but in some special cases those infinities can be consistently ploughed back into the theory to yield a finite end result with a small number of arbitrary constants remaining; these then being obtained from experiment [3, 4]. This is the renowned scheme of renormalization, disapproved of by some, but reasonably well-defined and yielding results in excellent agreement with Nature. The fact that only some theories are renormalizable has the beneficial effect of being selective, and so predictive. Unfortunately, in the usual sense, general relativity is *not* renormalizable [5].

The approach taken is to review the problem of the nonquantizability of traditional gravity, and to see it from the point of view of nonpredictability. Further physical input is then suggested to restore predictability. An illustrative one-loop calculation is then performed using the operator regularization technique.

2. Traditional formulation

Orthodox quantum gravity is a perturbatively unrenormalizable theory in the traditional sense, for starting from the example of a free scalar field in a gravitational field described by the classical Lagrangian in Euclidean space:

$$\mathcal{L} = -\sqrt{g}\left(R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2\right) \tag{1}$$

using units where $16\pi G = 1$, $c = 1$; one discovers, upon quantizing both the matter and gravitational fields, that the counter terms do not fall back within the original Lagrangian. Already at one loop one observes the appearance of ϕ^4 and p^4 counter terms (most easily seen by power counting), where p^2 is shorthand for $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, and not the independent variable of Hamiltonian mechanics. At two loops one also has such divergences, along with the occurrence of additional counter terms of ϕ^6 and p^6 form. This continues indefinitely, and since the total number of counter terms is then infinite in number, their associated ambiguities destroy the predictive power of the theory. The presence of higher derivative counter terms further destroys the causal behaviour of the theory. The above is the starting Lagrangian and *not* the classical Lagrangian that arises from the quantum theory in the $\hbar \rightarrow 0$ limit.

In summary, renormalizability requires that the number of independent counter terms be finite in number and that they do not spoil the physical behaviour of the original theory (modification is permitted). The fact that even after successful renormalization some factors, such as mass and charge, are left undetermined should perhaps not be viewed as a predictive shortcoming, since the fundamental units of nature are relative; that is to say, the choice of reference unit (be it mass, length, time, or charge) is always arbitrary, and then everything else can be stated in terms of these few units.

These observations motivate the consideration of the most general bare starting Lagrangian (in even powers of ϕ_0 and p_0) permitted on the grounds of symmetry:

$$\mathcal{L}_0 = -\sqrt{g_0}\left(\begin{array}{l} -2\Lambda_0 + R_0 + \frac{1}{2}p_0^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{1}{4!}\phi_0^4\lambda_0(\phi_0^2) + p_0^2\phi_0^2\kappa(\phi_0^2) + R_0\phi_0^2\gamma_0(\phi_0^2) \\ + p_0^4 a_0(p_0^2, \phi_0^2) + R_0 p_0^2 b_0(p_0^2, \phi_0^2) + R_0^2 c_0(p_0^2, \phi_0^2) + R_{0\mu\nu}R_0^{\mu\nu} d_0(p_0^2, \phi_0^2) + \ldots \end{array}\right) \tag{2}$$

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[1]R.P. Feynman. Lectures on gravitation. Caltech, 1962–1963, unpublished.

where p_0^2 is shorthand for $g_0^{\mu\nu}\partial_\mu\phi_0\partial_\nu\phi_0$ and λ_0, κ_0, γ_0, a_0, b_0, c_0, d_0, ... are arbitrary analytic functions. The second line carries all the higher derivative terms.

Strictly this is formal in having neglected gauge fixing and the resulting presence of ghost particles. Symmetry now assures us that all counter terms must fall back within this Lagrangian, and it is this that motivated the construction. So we are led to the renormalized theory given by

$$\mathcal{L} = -\sqrt{g}\left(\begin{array}{c} -2\Lambda + R + \frac{1}{2}p^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\phi^4\lambda(\phi^2) + p^2\phi^2\kappa(\phi^2) + R\phi^2\gamma(\phi^2) \\ +p^4a(p^2,\phi^2) + Rp^2b(p^2,\phi^2) + R^2c(p^2,\phi^2) + R_{\mu\nu}R^{\mu\nu}d(p^2,\phi^2) + \ldots \end{array}\right) \tag{3}$$

This expression has no predictive content, since there are an infinite number of arbitrary constants, in each arbitrary function, and in this sense the theory is not renormalized. However, there remain physical criteria to pin down some of these arbitrary factors.

The cosmological constant is abandoned on the grounds of energy conservation [6], but could just as well be retained. Since in general the higher derivative terms lead to acausal behaviour, their renormalized coefficient can also be put down to zero. This still leaves the three arbitrary analytic functions $\lambda(\phi^2)$, $\kappa(\phi^2)$, and $\gamma(\phi^2)$ associated with the terms ϕ^4, $p^2\phi^2$ and $R\phi^2$, respectively. The last may be abandoned on the grounds of defying the equivalence principle. To see this, begin by considering the first term of the Taylor expansion, namely $R\phi^2$; this has the form of a mass term and so one would be able to make a local measurement of mass to determine the curvature, and so contradict the equivalence principle. The same line of reasoning applies to the remaining terms, $R\phi^4$, $R\phi^6$,... etc.

This leaves us the two remaining infinite families of ambiguities within the terms $\phi^4\lambda(\phi^2)$ and $p^2\phi^2\kappa(\phi^2)$. In the limit of flat space in 3 + 1 dimensions this will reduce to a renormalized theory in the traditional sense if $\lambda(\phi^2)$ = constant, and $\kappa(\phi^2) = 0$. So one is led to proposing that the physical parameters should be

$$\begin{aligned} &\Lambda = \kappa(\phi^2) = \gamma(\phi^2) = 0 \\ &a(p^2,\phi^2) = b(p^2,\phi^2) = c(p^2,\phi^2) = d(p^2,\phi^2) = \ldots = 0 \\ &\lambda(\phi^2) = \lambda = \text{scalar particle self-coupling constant} \\ &m = \text{mass of the scalar particle} \end{aligned} \tag{4}$$

and so the renormalized theory of quantum gravity for a scalar field should have the form

$$\mathcal{L} = -\sqrt{g}\left(R + \frac{1}{2}p^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4\right) \tag{5}$$

One might worry about the renomalization group pulling the coupling constants around. This is an open point to which I feel one of three things might happen.

(1) The couplings, set to zero at a low energy scale, might reappear around the Plank scale. Whether the resulting theory then makes sense is a matter for dispute.

(2) Certain extra coupling constants (beyond those already set to zero) should be related, in order that the associated beta functions be zero (a fixed point), ensuring that all the couplings set to zero stay zero. This consistency condition could be the basis of a unification scheme.

(3) Some work [7, 8] suggests that for traditionally unrenormalized theories a consistency condition arises that fixes the renormalization group parameter, supposedly at the Plank scale for gravity. The idea is very tentative and not conclusive.

This is a matter that needs looking at more closely.

We are left with a finite theory that has few arbitrary constants. Despite the patchwork line of reasoning invoked to arrive at this hypothesis, one might alter perspective and simply be interested in investigating the consequences of such a scheme for its own sake, where many of the arbitrary factors are set to zero, for whatever reason. At this stage, any well-behaved, finite theory is worth investigating; and it is unfortunate that we do not have the guiding hand of mother nature to assist us in the guessing game.

3. Visable formulation

Having discussed this approach within the context of traditional renormalization, it is intriguing to note that the use of analytic continuation [9–13], and the more recent method of operator regularization [14–21] implements the above scheme in a much cleaner way.

In operator regularization one normally removes divergences using the analytical continuation

$$\Omega^{-m} = \lim_{\varepsilon\to 0}\frac{d^n}{d\varepsilon^n}\left(\frac{\varepsilon^n}{n!}\Omega^{-\varepsilon-m}\right) \tag{6}$$

where n is chosen sufficiently large to eliminate the infinites. This is explicitly illustrated later through an example. In this limited form, it is in effect just an automated method of minimal subtraction.

The ambiguities that were present before, but seen missing here, are actually resident in the general operator regularization formula

$$\Omega^{-m} = \lim_{\varepsilon\to 0}\frac{d^n}{d\varepsilon^n}\left((1+\alpha_1\varepsilon+\ldots+\alpha_n\varepsilon^n)\frac{\varepsilon^n}{n!}\Omega^{-\varepsilon-m}\right) \tag{7}$$

the alphas being ambiguous.

This identity is easily confirmed to be a valid rendition of Ω^{-m}, just as was the original. However, without any criterion to fix the ambiguities, we are obliged to carry them along. One might question having stopped the added series at ε^n, and this is understood when one recalls that n was chosen large enough to cancel any divergence in $\Omega^{-\varepsilon-m}$. The proposed further term would then have a contribution of the form

$$\Omega^{-m} = \lim_{\varepsilon\to 0}\frac{d^n}{d\varepsilon^n}\left(\alpha_{n+1}\varepsilon^{n+1}\underbrace{\frac{\varepsilon^n}{n!}\Omega^{-\varepsilon-m}}_{\text{finite}}\right) \tag{8}$$

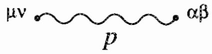
which is strictly zero.

Proceeding by adopting the above general operator regularization formula, one would deal with the ambiguities (alphas) as before, setting most of the final renormalized parameters to zero on physical grounds. The method of operator regularization has the strength of explicitly maintaining invariances, as well as being applicable to all loop levels, unlike the original Zeta function technique [11–13] that only applied to one loop.

To see this method in action and its calculational similarity to the dimensional regularization method,[2] we will walk through a simple example of a divergent one-loop diagram of a massive scalar particle in quantum gravity. So begin with an investigation of a massive scalar theory in its own induced gravitational field, described by the Lagrangian in Euclidean space:

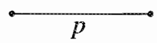
$$\mathcal{L} = -\sqrt{g}\left(R + \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2} m^2\phi^2 + \frac{1}{4!}\lambda\phi^4\right) \tag{9}$$

The Euclidean Feynman rules (of which there are an infinite number) we explicitly list, the gauged graviton propagator being derived from the gravitational, R, Lagrangian (Les Houches XXVIII) [5] in the harmonic gauge:



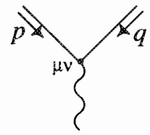
$$\frac{\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta}}{p^2} \tag{10}$$

Scalar propagator:



$$\frac{1}{p^2 + m^2} \tag{11}$$

First interaction vertex:



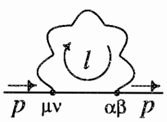
$$\frac{1}{2}\left(\delta_{\mu\nu}(p\cdot q - m^2) - p_\mu q_\nu - q_\mu p_\nu\right) \tag{12}$$

etc. using units where $\hbar = 1$.

Although there are an infinite number of Feynman diagrams, only a finite number are used to any finite loop order.

4. Divergent one-loop diagram example

Set about a one-loop investigation with matter particles on the external legs



$$= \int_{-\infty}^{\infty}\frac{d^4 l}{(2\pi)^4}\left(\frac{1}{l^2+m^2}\right)\left(\frac{\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta}}{(l+p)^2}\right)\frac{1}{2}\left(\delta_{\mu\nu}(p\cdot l - m^2) - p_\mu l_\nu - l_\mu p_\nu\right) \times \frac{1}{2}\left(\delta_{\alpha\beta}(p\cdot l - m^2) - p_\alpha l_\beta - l_\alpha p_\beta\right) \tag{13}$$

expand out the indices to yield

$$= \int_{-\infty}^{\infty}\frac{d^4 l}{(2\pi)^4}\frac{1}{l^2+m^2}\frac{1}{(l+p)^2}(p^2 l^2 + 2m^2 p\cdot l - 2m^4) \tag{14}$$

Then introduce the standard Feynman parameter "trick"

$$\frac{1}{D_1^{a_1}D_2^{a_2}\dots D_k^{a_k}} = \frac{\Gamma(a_1+a_2+\dots a_k)}{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_k)}\int_0^1\dots\int_0^1 dx_1\dots dx_k\frac{\delta(1-x_1-\dots x_k)x_1^{a_1-1}\dots x_k^{a_k-1}}{(D_1x_1+\dots D_kx_k)^{a_1+\dots a_k}} \tag{15}$$

to yield

$$= \int_{-\infty}^{\infty}\frac{d^4 l}{(2\pi)^4}\int_0^1 dx\frac{p^2l^2 + 2m^2 p\cdot l - 2m^4}{[l^2 + m^2x + p^2(1-x) + 2l\cdot p(1-x)]^2} \tag{16}$$

Remove divergences using the analytic continuation

$$\Omega^{-m} = \lim_{\varepsilon\to 0}\frac{d^n}{d\varepsilon^n}\left((1+\alpha_1\varepsilon+\dots+\alpha_n\varepsilon^n)\frac{\varepsilon^n}{n!}\Omega^{-\varepsilon-m}\right) \tag{17}$$

n being chosen sufficiently large to cancel the infinities. For the case in hand, $n = 1$ is adequate:

$$\Omega^{-2} = \lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}((1+\alpha\varepsilon)\varepsilon\Omega^{-\varepsilon-2}) \tag{18}$$

This yields

$$= \int_0^1 dx\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\int_{-\infty}^{\infty}\frac{d^4 l}{(2\pi)^4}\left(\varepsilon(1+\alpha\varepsilon)\frac{p^2l^2 + 2m^2p\cdot l - 2m^4}{[l^2+m^2x+p^2(1-x)+2l\cdot p(1-x)]^{\varepsilon+2}}\right) \tag{19}$$

Then performing the momentum integrations using [4]

$$\int_{-\infty}^{\infty}\frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{1}{(l^2+M^2+2l\cdot p)^A} = \frac{1}{(4\pi)^\omega\Gamma(A)}\frac{\Gamma(A-\omega)}{(M^2-p^2)^{A-\omega}} \tag{20}$$

$$\int_{-\infty}^{\infty}\frac{d^{2\omega}l}{(2\pi)^{2\omega}}\frac{l_\mu}{(l^2+M^2+2l\cdot p)^A} = -\frac{1}{(4\pi)^\omega\Gamma(A)}p_\mu\frac{\Gamma(A-\omega)}{(M^2-p^2)^{A-\omega}} \tag{21}$$

[2] The choice of the symbol ε was not by accident.

$$\int_{-\infty}^{\infty} \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \frac{l_\mu l_\nu}{(l^2+M^2+2l\cdot p)^A} = \frac{1}{(4\pi)^\omega \Gamma(A)} \left[p_\mu p_\nu \frac{\Gamma(A-\omega)}{(M^2-p^2)^{A-\omega}} + \frac{\delta_{\mu\nu}}{2} \frac{\Gamma(A-\omega-1)}{(M^2-p^2)^{A-\omega-1}} \right] \tag{22}$$

yields the finite expression

$$= \frac{1}{(4\pi)^2} \int_0^1 dx \lim_{\varepsilon\to 0} \frac{d}{d\varepsilon} \left(\frac{\varepsilon(1+\alpha\varepsilon)}{\Gamma(\varepsilon+2)} \left(\frac{p^4(1-x)^2\Gamma(\varepsilon)}{[m^2x+p^2x(1-x)]^\varepsilon} + 2\frac{p^2\Gamma(\varepsilon-1)}{[m^2x+p^2x(1-x)]^{\varepsilon-1}} - 2\frac{m^2p^2(1-x)\Gamma(\varepsilon)}{[m^2x+p^2x(1-x)]^\varepsilon} - 2\frac{m^4\Gamma(\varepsilon)}{[m^2x+p^2x(1-x)]^\varepsilon} \right)\right) \tag{23}$$

Doing the ε differential using

$$\lim_{\varepsilon\to 0} \frac{d}{d\varepsilon} \left(\frac{\varepsilon(1+\alpha\varepsilon)}{\Gamma(\varepsilon+2)} \left(a\frac{\Gamma(\varepsilon)}{\chi^\varepsilon} + b\frac{\Gamma(\varepsilon-1)}{\chi^{\varepsilon-1}} \right)\right) = -a + (a-b\chi)(\alpha - \ln(\chi)) \tag{24}$$

yields

$$= \frac{1}{(4\pi)^2} \int_0^1 dx \left(((2m^4+2m^2p^2-p^4)+p^4x(4-3x))(\ln(m^2x+p^2x(1-x))-\alpha) + 2m^4+2m^2p^2-p^4-p^2x(2m^2-2p^2+p^2x) \right) \tag{25}$$

and finally performing the x integration gives rise to the final result in Euclidean space:

$$= \frac{m^4}{(4\pi)^2} \left(\left(3+2\frac{p^2}{m^2}+\frac{m^2}{p^2}\right)\ln\left(1+\frac{p^2}{m^2}\right) - 1 - \frac{5}{2}\frac{p^2}{m^2} - \frac{1}{6}\frac{p^4}{m^4} + 2\left(1+\frac{p^2}{m^2}\right)\left(\ln\left(\frac{m^2}{\mu^2}\right)-\alpha\right)\right) \tag{26}$$

where there is no actual divergence at $p = 0$, and it should be commented that the use of a computer mathematics package can in general greatly reduced "calculator" fatique. The factor μ appears on dimensional grounds.

In this one-loop example the ambiguity α could be carried in the equally arbitrary renormalization group parameter μ. This cannot continue to be so at higher loop order where further ambiguities will arise that then cannot be taken up in μ.

5. Conclusions

Traditional gravity is not quantizable because in the attempt to renormalize it, all predictability is lost. However, it is proposed that by the application of further physical conditions, predictability can be restored. Possible undesirable behaviour of this theory at the Plank scale is seen either as a failing or as a driving force for unification.

Although the proposed scheme can be implemented under many regularization schemes, preference is given to the analytic continuation technique, although in a generalized form. An example one-loop calculation is seen to be finite, not requiring any manual subtractions of divergences.

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