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Rigorous formulation of duality in gravitational theories

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Abstract

In this paper we evince a rigorous formulation of duality in gravitational theories where an Einstein-like equation is valid, by providing the conditions under which $\underset{g}{\star}\mathcal{T}^{\alpha}$ and $\underset{g}{\star}\mathcal{R}^{\alpha}_{\beta}$ may be considered as the torsion and curvature 2-forms associated with a connection D', part of a Riemann–Cartan structure $(M, \boldsymbol{g}', D')$, in the cases $\boldsymbol{g}' = \boldsymbol{g}$ and $\boldsymbol{g}' \neq \boldsymbol{g}$, once $\mathcal{T}^{\alpha}$ and $\mathcal{R}^{\alpha}_{\beta}$ are the torsion and curvature 2-forms associated with a connection D part of a Riemann–Cartan structure $(M, \boldsymbol{g}, D)$. A new form for the Einstein equation involving the dual of the Riemann tensor of D is also provided, and the result is compared with others appearing in the literature.

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1. Introduction

There have been a number of papers trying to provide evidence of a possible analogy between electromagnetism and gravitation, in order to elicit a gravitational analog for the magnetic monopole that appears in the generalized Maxwell equations with magnetic and electric currents[3]. Some of these earlier papers are in[4] [10, 19, 20]. Ten years ago Nieto [21] developed an analog of S-duality[5] for linearized gravity in (3 + 1) dimensions (see also [14, 15, 22])

[3] In such theory, see, e.g., [17, 29] which uses two potentials, the electric and magnetic currents are phenomenological, i.e. the magnetic current is not a result of a $U(1)$ gauge theory formulated in a nontrivial base spacetime. So, in the theory which uses two potentials there are no Dirac strings at all. Unfortunately, this result is sometimes overlooked in presentations of the monopole theory and in the proposed gravitational analogies of that concept.
[4] One of the motivations of [10] was eventually to obtain a quantization of mass.
[5] Duality and S-duality have been also studied extensively in non-Abelian gauge theories, see, e.g. [18, 24, 31] and references therein.

and generalizations of that idea of *duality* for gravitational theories in more dimensions appear, e.g., in [1–3, 5–7, 12, 13]. In particular for the case of gravity in (3 + 1) dimensions, a set of equations have been proposed for Einstein equations, Bianchi identities and their duals, although mainly used in the linear approximation.

The main aim of this note is to derive exact equations that must be satisfied by the dual Einstein equations and for the duals of the torsion and curvature 2-forms of a general Riemann–Cartan structure $(M, \boldsymbol{g}, D)$. We study in which conditions the dualized objects realize a Riemann–Cartan structure $(M, \boldsymbol{g}, D')$, or even a $(M, \boldsymbol{g}', D')$ one. In doing so we find that the correct field equations for a dual theory (in a precise mathematical sense defined below) are at variance with ones proposed in some of the above mentioned papers. In doing so we hope that the present note be useful for those pursuing the interesting ideas of duality in gravitational theories.

The paper, which uses an intrinsic formulation of the theories presented, is organized as follows. In section 2 we present some necessary preliminaries that serve, besides the proposal of introducing our notation, also for the purpose of presenting what it is understood here by a Riemann–Cartan gravitational theory. In this section we also review the Bianchi identities for the torsion and curvature 2-forms $\mathcal{T}^\alpha$ and $\mathcal{R}^\alpha_\beta$ of $(M, \boldsymbol{g}, D)$ in intrinsic and component forms, because those formulas for a Riemann–Cartan theory are not well known as they deserve to be, and sometimes concealed from the formalism. In section 3 we introduce the Ricci 1-form fields $\mathcal{R}^\mu$ and the Einstein 1-forms fields $\mathcal{G}^\mu$ [27], and further prove a proposition containing a formula that relates the dual $\underset{\boldsymbol{g}}{\star}\mathcal{R}^\mu$ of $\mathcal{R}^\mu$ to a sum, involving the *dual* of the Riemann tensor and an important formula for the dual $\underset{\boldsymbol{g}}{\star}\mathcal{G}^\mu$ of $\mathcal{G}^\mu$, that permits us to write Einstein equations in a suggestive way concerning duality structures. In section 4 we provide the correct dual of Einstein equation in Riemann–Cartan theory. In section 5 we delve into the formalism under which conditions $\underset{\boldsymbol{g}}{\star}\mathcal{T}^\alpha$ and $\underset{\boldsymbol{g}}{\star}\mathcal{R}^\alpha_\beta$ may be considered as the torsion and curvature 2-forms associated with a connection D' part of a Riemann–Cartan structure $(M, \boldsymbol{g}, D')$. Our result is then compared in section 6 with the ones, e.g., in [1], which are then commented and analyzed in the present context. In section 7 we study the same problem as in section 5 but this time asking the conditions under which $\underset{\boldsymbol{g}}{\star}\mathcal{T}^\alpha$ and $\underset{\boldsymbol{g}}{\star}\mathcal{R}^\alpha_\beta$ may be considered as the torsion and curvature 2-forms associated with a connection D' part of a Riemann–Cartan structure $(M, \boldsymbol{g}', D')$ with $\boldsymbol{g}' \neq \boldsymbol{g}$. In section 8 we present our conclusions. The paper contains some appendices reviewing the definition of the exterior covariant derivative of *indexed* form fields, the decomposition of the Riemann and Ricci tensors of a general Riemann–Cartan structure $(M, \boldsymbol{g}, D)$, together with their respective similars for a Lorentzian structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$, needed to perceive some statements in the main text. There is also an appendix containing a collection of identities involving the contraction of differential forms and Hodge duals used in the derivations hereon.

2. Some necessary preliminaries

We start this section by recalling some germane facts concerning the Riemann–Cartan structures and a particular and outstanding case of those structures, the Lorentzian one, which serves for the purpose of fixing our notations, besides other relevant properties and prominent applications. In what follows a general Riemann–Cartan structure will be denoted by $(M, \boldsymbol{g}, D)$. Here M is a four-dimensional Hausdorff, paracompact, connected and noncompact manifold, $\boldsymbol{g} \in \sec T^0_2 M$ a metric tensor field of signature (1, 3), D is an affine connection [8, 9, 11] on M. Also the connection D is metric compatible, i.e. $D\boldsymbol{g} = 0$

and, moreover, for a general Riemann–Cartan structure the torsion and curvature tensors[6] of D—denoted by $\mathcal{T}$ and $\mathbf{R}$—are non null. When $\mathcal{T} = 0$ and $\mathbf{R} \neq 0$, a Riemann–Cartan structure is called a *Lorentzian structure* and will be denoted by $(M, \boldsymbol{g}, \mathring{D})$.[7] When $\mathbf{R} = 0$ a Lorentzian structure is called the Minkowski structure. To present the definition of $\mathcal{T}$ and $\mathbf{R}$, and the conventions used in this paper, first the torsion and curvature operations are introduced.

Definition 1. *Let* $\mathbf{u}, \mathbf{v} \in \sec TM$. *The torsion and curvature operations of a given affine connection [9] D are respectively the mappings:* $\boldsymbol{\tau} : \sec TM \otimes \sec TM \to \sec TM$ *and* $\boldsymbol{\rho} : \sec TM \otimes \sec TM \to \operatorname{End}(\sec TM)$ *given by*

$$\tau(\mathbf{u}, \mathbf{v}) = D_{\mathbf{u}}\mathbf{v} - D_{\mathbf{v}}\mathbf{u} - [\mathbf{u}, \mathbf{v}], \tag{1}$$

$$\rho(\mathbf{u}, \mathbf{v}) = D_{\mathbf{u}} D_{\mathbf{v}} - D_{\mathbf{v}} D_{\mathbf{u}} - D_{[\mathbf{u},\mathbf{v}]}. \tag{2}$$

Definition 2. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \sec TM$ *and* $\alpha \in \sec \Lambda^1 T^*M$. *The torsion and curvature tensors of an affine connection D are the mappings [8, 9, 11]* $\mathcal{T} : \sec(\Lambda^1 T^*M \otimes TM \otimes TM) \to \mathcal{F}(M)$ *and* $\mathbf{R} : \sec(TM \otimes \Lambda^1 T^*M \otimes TM \otimes TM) \to \mathcal{F}(M)$ *given by*

$$\mathcal{T}(\alpha, \mathbf{u}, \mathbf{v}) = \alpha(\tau(\mathbf{u}, \mathbf{v})), \tag{3}$$

$$\mathbf{R}(\mathbf{w}, \alpha, \mathbf{u}, \mathbf{v}) = \alpha(\rho(\mathbf{u}, \mathbf{v})\mathbf{w}), \tag{4}$$

where $\mathcal{F}(M)$ *is the set of functions on* M.

Given an arbitrary moving frame $\{\boldsymbol{e}_\alpha\}$ on TM, let $\{\theta^\rho\}$ be the *dual frame of* $\{\boldsymbol{e}_\alpha\}$ (i.e. $\theta^\rho(\boldsymbol{e}_\alpha) = \delta^\rho_\alpha$). Let also $\{\boldsymbol{e}^\alpha\}$ be the reciprocal basis of $\{\boldsymbol{e}_\beta\}$, i.e., $\boldsymbol{g}(\boldsymbol{e}^\alpha, \boldsymbol{e}_\beta) = \delta^\alpha_\beta$ and let $\{\theta_\alpha\}$ be the reciprocal basis of $\{\theta^\rho\}$, i.e., $\theta_\alpha(\boldsymbol{e}^\beta) = \delta^\beta_\alpha$. We write

$$[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = c^\rho_{\alpha\beta}\boldsymbol{e}_\rho, \qquad D_{\boldsymbol{e}_\alpha}\boldsymbol{e}_\beta = L^\rho_{\alpha\beta}\boldsymbol{e}_\rho, \tag{5}$$

where $c^\rho_{\alpha\beta}$ are the *structure coefficients* of the frame $\{\boldsymbol{e}_\alpha\}$ and $L^\rho_{\alpha\beta}$ are the *connection coefficients* in this frame. Then, the components of the torsion and curvature tensors are given, respectively, by

$$\begin{aligned} &\mathcal{T}(\theta^\alpha, \boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) = T^\rho_{\alpha\beta} = L^\rho_{\alpha\beta} - L^\rho_{\beta\alpha} - c^\rho_{\alpha\beta} \\ &\mathbf{R}(\boldsymbol{e}_\mu, \theta^\alpha, \boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) = R_\mu{}^\rho{}_{\alpha\beta} = \boldsymbol{e}_\alpha\left(L^\rho_{\beta\mu}\right) - \boldsymbol{e}_\beta\left(L^\rho_{\alpha\mu}\right) + L^\rho_{\alpha\sigma}L^\sigma_{\beta\mu} - L^\rho_{\beta\sigma}L^\sigma_{\alpha\mu} - c^\sigma_{\alpha\beta}L^\rho_{\sigma\mu}. \end{aligned} \tag{6}$$

We can easily verify that defining

$$R_{\mu\nu\alpha\beta} := g_{\mu\rho}R_\mu{}^\rho{}_{\alpha\beta} \tag{7}$$

it follows that

$$R_{\mu\nu\alpha\beta} = R_{\nu\mu\alpha\beta} = R_{\mu\nu\beta\alpha}. \tag{8}$$

Remark 3. When the torsion tensor of D is null, besides the symmetries given in equation (8), the symmetry

$$R_{\mu\nu\alpha\beta} = R_{\beta\alpha\mu\nu} \tag{9}$$

also holds.

[6] For the conventions used for those tensors in this paper see the appendix.

[7] The connection satisfying $\mathring{D}\boldsymbol{g} = 0$ and $\mathcal{T} = 0$ is unique and is called the Lévi-Cività connection of $\boldsymbol{g}$.

Now, taking into account equation (8) we introduce also a 'physically equivalent' Riemann tensor $\boldsymbol{R}$ by

$$\begin{aligned}\boldsymbol{R} &= \tfrac{1}{4}R_{\mu\nu\alpha\beta}\theta^{\mu}\wedge\theta^{\nu}\otimes\theta^{\alpha}\wedge\theta^{\beta} = \tfrac{1}{4}R^{\mu\nu}{}_{\alpha\beta}\theta_{\mu}\wedge\theta_{\nu}\otimes\theta^{\alpha}\wedge\theta^{\beta}\\ &= \tfrac{1}{4}R_{\mu\nu}^{\ \ \alpha\beta}\theta^{\mu}\wedge\theta^{\nu}\otimes\theta_{\alpha}\wedge\theta_{\beta}.\end{aligned} \tag{10}$$

In addition,

$$\mathrm{d}\theta^{\rho} = -\tfrac{1}{2}c^{\rho}_{\alpha\beta}\theta^{\alpha}\wedge\theta^{\beta}, \qquad D_{e_{\alpha}}\theta^{\rho} = -L^{\rho}_{\alpha\beta}\theta^{\beta} \tag{11}$$

where $\omega^{\rho}_{\beta}\in\sec\Lambda^{1}T^{*}M$ given by

$$\omega^{\rho}_{\beta} = L^{\rho}_{\alpha\beta}\theta^{\alpha}, \tag{12}$$

are the so-called *connection 1-forms* [8, 11] *relative to the cobasis* $\{\theta^{\alpha}\}$. *Moreover, the* $\mathcal{T}^{\rho}\in\sec\Lambda^{2}T^{*}M$ are the *torsion 2-forms* and the $\mathcal{R}^{\rho}_{\beta}\in\sec\Lambda^{2}T^{*}M$ are the *curvature 2-forms* [8, 11], given respectively by

$$\mathcal{T}^{\rho} = \tfrac{1}{2}T^{\rho}_{\alpha\beta}\theta^{\alpha}\wedge\theta^{\beta}, \qquad \mathcal{R}^{\rho}_{\mu} = \tfrac{1}{2}R_{\mu}{}^{\rho}{}_{\alpha\beta}\theta^{\alpha}\wedge\theta^{\beta}. \tag{13}$$

Multiplying equations (6) by $\frac{1}{2}\theta^{\alpha}\wedge\theta^{\beta}$ and using equations (11) and (13), *Cartan's structure equations* are derived:

$$\mathcal{T}^{\rho} = \mathrm{d}\theta^{\rho} + \omega^{\rho}_{\beta}\wedge\theta^{\beta}, \qquad \mathcal{R}^{\rho}_{\mu} = \mathrm{d}\omega^{\rho}_{\mu} + \omega^{\rho}_{\beta}\wedge\omega^{\beta}_{\mu}. \tag{14}$$

Definition 4. *A Riemann–Cartan spacetime is a pentuple* $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ *where* $(M, \boldsymbol{g}, D)$ *is a Riemann–Cartan structure, and we suppose the existence of a global* $\tau_{\boldsymbol{g}}\in\sec\Lambda^{4}T^{*}M$ *(which as well known defines an orientation for M). Moreover,* $\uparrow$ *denotes that the Riemann–Cartan structure is time oriented. See, e.g., [25, 27] for details.*

Pentuples $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ represent gravitational fields in the so-called Riemann–Cartan theories. In the theory presented, e.g., in [16], the equations of motion are the Einstein equation,

$$\mathbf{G} = \mathbf{T}, \tag{15}$$

where $\mathbf{G}\in\sec T^{0}_{2}M$ is the Einstein tensor, $\mathbf{T}\in\sec T^{0}_{2}M$ is the *canonical energy–momentum tensor* of the matter fields, and the algebraic identity

$$\boldsymbol{\Upsilon}_{\alpha\beta} = \mathbf{J}_{\alpha\beta}, \tag{16}$$

where the $\boldsymbol{\Upsilon}_{\alpha\beta}\in\sec\Lambda^{1}T^{*}M$ are such that their components are the so-called modified torsion *tensor components*, and the $\underset{\boldsymbol{g}}{\star}\mathbf{J}_{\alpha\beta}\in\sec\Lambda^{3}T^{*}M$ are the spin angular momentum densities of the matter fields[8]. Also, the symbol $\underset{\boldsymbol{g}}{\star}$ denotes the Hodge star operator associated with the metric $\boldsymbol{g}$.

Remark 5. It is crucial to observe that for a general Riemann–Cartan structure, $\mathbf{G} = G_{\mu\nu}\theta^{\mu}\otimes\theta^{\nu}$ and $\mathbf{T} = T_{\mu\nu}\theta^{\mu}\otimes\theta^{\nu}$ are not symmetric, i.e. $G_{\mu\nu}\neq G_{\nu\mu}$ and $T_{\mu\nu}\neq T_{\nu\mu}$. We recall that

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R, \tag{17}$$

[8] The components of $\mathbf{J}_{\alpha\beta}\in\sec\Lambda^{1}T^{*}M$ are the standard (field theory) canonical spin angular momentum of the matter fields. In the Riemann–Cartan theory of [16], since equation (16) is an algebraic identity, it is possible to eliminate completely the torsion tensor from the theory and to write an Einstein equation involving the Einstein tensor of the Lévi-Cività connection of $\boldsymbol{g}$ (using the decomposition presented in appendix B) and a metric energy–momentum tensor that is equivalent to the Belinfante symmetrization of the canonical energy–momentum tensor of the theory.

where $R_{\mu\nu}$ are the components of the Ricci tensor (which, as $G_{\mu\nu}$ are not symmetric)

$$Ricci = R_{\mu\nu}\theta^{\mu} \otimes \theta^{\nu} := R_{\mu}{}^{\rho}{}_{\rho\nu}\theta^{\mu} \otimes \theta^{\nu}, \tag{18}$$

and $R = g^{\mu\nu} R_{\mu}{}^{\rho}{}_{\rho\nu}$ is the curvature scalar.

It is also well known that in *GRT* a gravitational field generated by a given matter distribution (represented by a given energy–momentum tensor $\mathring{\mathbf{T}} \in \sec T_2^0 M)$ is represented by a pentuple $(M, \mathring{\boldsymbol{g}}, \mathring{D}, \tau_{\mathring{\boldsymbol{g}}}, \uparrow)$[9] and the equation of motion (Einstein equation) is given by

$$\mathring{\mathbf{G}} = \mathring{T} \tag{19}$$

and in this case the tensors $\mathring{\mathbf{G}}$ and $\mathring{\mathbf{T}}$ are symmetric.

Remark 6. In the appendix we review how to write the Riemann curvature tensor (respectively the Einstein tensor) of a Riemann–Cartan structure $(M, \boldsymbol{g}, D)$ in terms of the Riemann curvature tensor (respectively the Einstein tensor) of a Lorentzian structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$. Those results are important for a proper understanding of this paper.

2.1. The Bianchi identities

Given a general Riemann–Cartan structure $(M, \boldsymbol{g}, D)$ we have the following identities:

$$\mathbf{D}\mathcal{T}^{\alpha} = \mathcal{R}^{\alpha}_{\beta} \wedge \theta^{\beta}, \tag{20}$$

$$\mathbf{D}\mathcal{R}^{\alpha}_{\beta} = 0, \tag{21}$$

known respectively as the first and second Bianchi identities (see, e.g., [8, 27]). In the above equations, $\mathbf{D}$ is the exterior covariant derivative of indexed form fields [4, 27], whose precise definition is recalled in appendix A. Now, the coordinate expressions of equations (20) and (21) can be easily found and are respectively [8, 28] written as

$$\sum_{(\mu\alpha\beta)} R_{\mu}{}^{\rho}{}_{\alpha\beta} = \sum_{(\mu\alpha\beta)} \left(D_{\mu} T^{\rho}_{\alpha\beta} - T^{\kappa}_{\mu\beta} T^{\rho}_{\kappa\alpha}\right), \tag{22}$$

$$\sum_{(\mu\nu\rho)} D_{\mu} R^{\ \alpha}_{\beta\ \nu\rho} = \sum_{(\mu\nu\rho)} T^{\kappa}_{\nu\mu} R^{\ \alpha}_{\beta\ \kappa\rho}, \tag{23}$$

where $\sum_{(\mu\nu\rho)}$ denotes (as usual) the cyclic sum. For future use we observe that

$$\mathcal{R}^{\alpha}_{\beta} \wedge \theta^{\beta} = \frac{1}{3!}\left(R_{\mu}{}^{\alpha}{}_{\alpha\beta} + R_{\alpha}{}^{\alpha}{}_{\beta\mu} + R_{\beta}{}^{\alpha}{}_{\mu\alpha}\right)\theta^{\mu} \wedge \theta^{\alpha} \wedge \theta^{\beta}. \tag{24}$$

Remark 7. For a Lorentz structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$ the Bianchi identities reduce to

$$\mathring{\mathcal{R}}^{\alpha}_{\beta} \wedge \theta^{\beta} = 0, \qquad \mathbf{D}\mathring{\mathcal{R}}^{\alpha}_{\beta} = 0,$$

or in components:

$$\sum_{(\mu\alpha\beta)} R_{\mu}{}^{\rho}{}_{\alpha\beta} = 0, \qquad \sum_{(\mu\nu\rho)} D_{\mu} R^{\ \alpha}_{\beta\ \nu\rho} = 0.$$

[9] In fact a gravitational field is defined by an equivalence class of pentuples, where $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ and $(M', \boldsymbol{g}', D', \tau'_{\boldsymbol{g}}, \uparrow')$ are said to be equivalent if there is a diffeomorphism $\mathtt{h} : M \to M'$, such that $\boldsymbol{g}' = \mathtt{h}^*\boldsymbol{g}$, $D' = \mathtt{h}^* D$, $\tau'_{\boldsymbol{g}} = \mathtt{h}^*\tau_{\boldsymbol{g}}, \uparrow' = \mathtt{h}^* \uparrow$, (where $\mathtt{h}^*$ here denotes the pullback mapping). For more details, see, e.g., [25, 27]. With the above definition we exclude from our considerations models with closed timelike curves, which according to our view are pure science fiction.

3. Ricci and Einstein 1-form fields

Given $R_{\mu\nu}$ and $G_{\mu\nu}$, respectively the components of the Ricci and Einstein tensors (in the general basis introduced above) we define the Ricci ($\mathcal{R}^\mu \in \sec \Lambda^1 T^*M$) and the Einstein ($\mathcal{G}^\mu \in \sec \Lambda^1 T^*M$) 1-form fields by[10]

$$\mathcal{R}^\mu := R^\mu_\nu \theta^\nu, \qquad \mathcal{G}^\mu := G^\mu_\nu \theta^\nu.$$

For future use we introduce also the energy–momentum 1-form fields $\mathbf{T}^\mu \in \sec \Lambda^1 T^*M$ by

$$\mathbf{T}^\mu := T^\mu_\nu \theta^\nu. \tag{25}$$

Also

$$\begin{aligned} \underset{g}{\star} \mathbf{T}^\mu &= T^\mu_\nu \underset{g}{\star} \theta^\nu \\ &= \frac{1}{3!}\left(T^\mu_\nu \sqrt{|\det \boldsymbol{g}|} g^{\nu\kappa} \epsilon_{\kappa\iota\lambda\sigma}\right)\theta^\iota \wedge \theta^\kappa \wedge \theta^\sigma. \end{aligned} \tag{26}$$

Proposition 8. [11]*The dual of the Ricci and Einstein 1-form fields, i.e.* $\underset{g}{\star}\mathcal{R}^\alpha \in \sec \Lambda^3 T^*M$ *and* $\underset{g}{\star}\mathcal{G}^\alpha \in \sec \Lambda^3 T^*M$, *can be written as*

$$\underset{g}{\star} \mathcal{R}^\alpha = -\underset{g}{\star} \mathcal{R}^\alpha_\beta \wedge \theta^\beta = -\theta^\beta \wedge \underset{g}{\star}\mathcal{R}^\alpha_\beta, \tag{27}$$

$$\underset{g}{\star} \mathcal{G}^\rho = -\tfrac{1}{2}\mathcal{R}_{\alpha\beta} \wedge \underset{g}{\star}(\theta^\alpha \wedge \theta^\beta \wedge \theta^\rho). \tag{28}$$

where $\mathcal{R}^\rho_\mu = \frac{1}{2} R_\mu{}^\rho{}_{\alpha\beta}\theta^\alpha \wedge \theta^\beta$ *and* $\mathcal{R}_{\mu\rho} := \frac{1}{2} R_{\mu\rho\alpha\beta}\theta^\alpha \wedge \theta^\beta$.

Proof. Using some of the identities in appendix C we can write immediately

$$\begin{aligned} \theta^\rho \wedge \underset{g}{\star}\mathcal{R}_{\mu\rho} &= -\underset{g}{\star}(\theta^\rho \underset{g}{\lrcorner} \mathcal{R}_{\mu\rho}) \\ &= -\underset{g}{\star} \tfrac{1}{2}[R_{\mu\rho\alpha\beta}\theta^\rho \underset{g}{\lrcorner}(\theta^\alpha \wedge \theta^\beta)] = -\underset{g}{\star}(R_{\mu\rho\alpha\beta} g^{\rho\alpha}\theta^\beta) \\ &= -\underset{g}{\star}\left(R_\mu{}^\alpha{}_{\alpha\beta}\theta^\beta\right) = -\star(R_{\mu\beta}\theta^\beta) \\ &= -\underset{g}{\star}\mathcal{R}_\mu, \end{aligned}$$

and equation (27) is proved.

Now equation (28) is evinced. By taking some of the identities in appendix C, we can immediately write

$$\begin{aligned} \tfrac{1}{2}\mathcal{R}_{\alpha\beta} \wedge \underset{g}{\star}(\theta^\alpha \wedge \theta^\beta \wedge \theta^\rho) &= -\tfrac{1}{2}\underset{g}{\star}[\mathcal{R}_{\alpha\beta} \underset{g}{\lrcorner}(\theta^\alpha \wedge \theta^\beta \wedge \theta^\rho)] \\ &= -\tfrac{1}{4} R_{\alpha\beta\iota\kappa} \underset{g}{\star}[(\theta^\iota \wedge \theta^\kappa)\underset{g}{\lrcorner}(\theta^\alpha \wedge \theta^\beta \wedge \theta^\rho)] \\ &= -\tfrac{1}{4} R_{\alpha\beta\iota\kappa} \underset{g}{\star}[(\theta^\iota \underset{g}{\lrcorner}(\theta^\kappa \underset{g}{\lrcorner}(\theta^\alpha \wedge \theta^\beta \wedge \theta^\rho))] \\ &= -\underset{g}{\star}\left(\mathcal{R}^\rho - \tfrac{1}{2}R\theta^\rho\right), \end{aligned}$$

and equation (28) is proved. □

[10] To the best of our knowledge the dual of the Ricci and Einstein 1-form fields first appear in [30]. See also [4, 11].
[11] Proofs of this proposition can also be found, e.g. [4, 11, 30]. The proof here seems a very simple one and it is given here for completeness of our exposition and benefit of the reader. Also in [26], using the Clifford bundle operator, it has been identified that the Ricci 1-forms can be obtained by application of the Ricci operator $\partial \wedge \partial$ to the 1-form fields θ^α, i.e. $(\partial \wedge \partial)\theta^\alpha = \mathcal{R}^\alpha$. Here $\partial = \theta^\kappa \mathring{D}_{\mathbf{e}_\kappa}$ is the Dirac operator [23] acting on sections of the Clifford bundle.

Remark 9. Recall that

$$\underset{g}{\star}\mathcal{R}_{\mu\rho} := \tfrac{1}{2}R_{\mu\rho\alpha\beta}\underset{g}{\star}(\theta^{\alpha}\wedge\theta^{\beta})$$
$$= \tfrac{1}{2}R_{\mu\rho\alpha\beta}\tfrac{1}{2}\sqrt{|\det\boldsymbol{g}|}g^{\alpha\kappa}g^{\beta\iota}\epsilon_{\kappa\iota\lambda\sigma}\theta^{\lambda}\wedge\theta^{\sigma} = \tfrac{1}{2}\big(\tfrac{1}{2}\sqrt{|\det\boldsymbol{g}|}\epsilon_{\kappa\iota\lambda\sigma}R_{\mu\rho}^{\ \ \kappa\iota}\big)\theta^{\lambda}\wedge\theta^{\sigma}$$
$$= \tfrac{1}{2}\mathsf{R}^{\star}_{\mu\rho\lambda\sigma}\theta^{\lambda}\wedge\theta^{\sigma}$$
$$= \tfrac{1}{2}R^{\star}_{\mu\rho\lambda\sigma}\sqrt{|\det\boldsymbol{g}|}\theta^{\lambda}\wedge\theta^{\sigma}, \tag{29}$$

with

$$\mathsf{R}^{\star}_{\mu\rho\lambda\sigma} := \tfrac{1}{2}\sqrt{|\det\boldsymbol{g}|}\epsilon_{\kappa\iota\lambda\sigma}R^{\kappa\iota}_{\ \ \alpha\beta}, \qquad \text{and} \qquad R^{\star}_{\mu\rho\lambda\sigma} := \tfrac{1}{2}\epsilon_{\kappa\iota\lambda\sigma}R^{\kappa\iota}_{\ \ \alpha\beta}, \tag{30}$$

and so it follows that

$$\star\mathcal{R}_{\mu\rho}\wedge\theta^{\rho} = \tfrac{1}{2}\mathsf{R}^{\star}_{\mu\rho\lambda\sigma}\theta^{\lambda}\wedge\theta^{\sigma}\wedge\theta^{\rho} = \tfrac{1}{2}\mathsf{R}^{\star}_{\mu\rho\lambda\sigma}\theta^{\rho}\wedge\theta^{\lambda}\wedge\theta^{\sigma}$$
$$= \tfrac{1}{2}\left(\tfrac{1}{3}\mathsf{R}^{\star}_{\mu\rho\lambda\sigma}\theta^{\rho}\wedge\theta^{\lambda}\wedge\theta^{\sigma} + \tfrac{1}{3}\mathsf{R}^{\star}_{\mu\lambda\sigma\rho}\theta^{\lambda}\wedge\theta^{\sigma}\wedge\theta^{\rho} + \tfrac{1}{3}\mathsf{R}^{\star}_{\mu\sigma\rho\lambda}\theta^{\sigma}\wedge\theta^{\rho}\wedge\theta^{\lambda}\right) \tag{31}$$

$$= \tfrac{1}{3!}\left(\mathsf{R}^{\star}_{\mu\rho\lambda\sigma} + \mathsf{R}^{\star}_{\mu\lambda\sigma\rho} + \mathsf{R}^{\star}_{\mu\sigma\rho\lambda}\right)\theta^{\rho}\wedge\theta^{\lambda}\wedge\theta^{\sigma}$$
$$= \tfrac{1}{3!}\left(R^{\star}_{\mu\rho\lambda\sigma} + R^{\star}_{\mu\lambda\sigma\rho} + R^{\star}_{\mu\sigma\rho\lambda}\right)\sqrt{|\det\boldsymbol{g}|}\theta^{\rho}\wedge\theta^{\lambda}\wedge\theta^{\sigma}, \tag{32}$$

and taking into account equation (27) it reads

$$\underset{g}{\star}\mathcal{R}_{\mu} = -\tfrac{1}{3!}\left(R^{\star}_{\mu\rho\lambda\sigma} + R^{\star}_{\mu\lambda\sigma\rho} + R^{\star}_{\mu\sigma\rho\lambda}\right)\sqrt{|\det\boldsymbol{g}|}\theta^{\rho}\wedge\theta^{\lambda}\wedge\theta^{\sigma}. \tag{33}$$

4. The dual of Einstein equation in Riemann–Cartan theory

We now return to equation (15) which in components can read[12]

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = T_{\mu\nu}. \tag{34}$$

Multiplying this equation on both sides by θ^{ν} and recalling the definitions of the Ricci, Einstein, and the energy–momentum 1-form fields given above we have

$$\mathcal{G}_{\mu} = \mathbf{T}_{\mu}. \tag{35}$$

Taking the dual of this equation we obtain

$$\underset{g}{\star}\mathcal{G}_{\mu} = \underset{g}{\star}\mathcal{R}_{\mu} - \tfrac{1}{2}R\underset{g}{\star}\theta_{\mu} = \underset{g}{\star}\mathbf{T}_{\mu} \tag{36}$$

Taking equations (33) and (26) into account, equation (36) can be expressed as

$$-\tfrac{1}{3!}\left(R^{\star}_{\mu\rho\lambda\sigma} + R^{\star}_{\mu\lambda\sigma\rho} + R^{\star}_{\mu\sigma\rho\lambda} + \tfrac{1}{2}R\delta^{\kappa}_{\mu}\epsilon_{\kappa\rho\lambda\sigma}\right)\sqrt{|\det\boldsymbol{g}|}\theta^{\rho}\wedge\theta^{\lambda}\wedge\theta^{\sigma}$$
$$= \tfrac{1}{3!}(T_{\mu\nu}g^{\nu\kappa}\epsilon_{\kappa\rho\lambda\sigma})\sqrt{|\det\boldsymbol{g}|}\theta^{\rho}\wedge\theta^{\lambda}\wedge\theta^{\sigma},$$

or equivalently

$$\left(R^{\star}_{\mu\rho\lambda\sigma} + R^{\star}_{\mu\lambda\sigma\rho} + R^{\star}_{\mu\sigma\rho\lambda} + \tfrac{1}{2}R\epsilon_{\mu\rho\lambda\sigma}\right) = \epsilon_{\rho\lambda\sigma\kappa}T_{\mu}^{\ \kappa}. \tag{37}$$

[12] Note that in equation (34) $R_{\mu\nu}$ and $T_{\mu\nu}$ are not symmetric.

4.1. The field and structure equations

We now summarize the field and Bianchi identities for a Riemann–Cartan theory where an Einstein-like equation holds. Those equations can be written *conveniently* in intrinsic and component forms respectively as

$$\underset{g}{\star}\mathcal{G}_\mu = \underset{g}{\star}\mathcal{R}_\mu - \frac{1}{2}R\underset{g}{\star}\theta_\mu = \underset{g}{\star}\mathbf{T}_\mu \tag{38}$$

$$\mathbf{D}\mathcal{T}^\alpha = \mathcal{R}^\alpha_\beta \wedge \theta^\beta, \tag{39}$$

$$\mathbf{D}\mathcal{R}^\alpha_\beta = 0, \tag{40}$$

$$\left(R^\star_{\mu\rho\lambda\sigma} + R^\star_{\mu\lambda\sigma\rho} + R^\star_{\mu\sigma\rho\lambda} + \frac{1}{2}R\epsilon_{\mu\rho\lambda\sigma}\right) = \epsilon_{\rho\lambda\sigma\kappa}T_\mu^{\ \kappa} \Leftrightarrow G_{\mu\nu} = T_{\mu\nu}, \tag{41}$$

$$\sum_{(\mu\alpha\beta)} R_{\mu\rho\alpha\beta} = \sum_{(\mu\alpha\beta)} \left(D_\mu T_{\rho\alpha\beta} - T^\kappa_{\mu\beta}T_{\rho\kappa\alpha}\right), \tag{42}$$

$$\sum_{(\mu\nu\rho)} D_\mu R_{\beta\alpha\nu\rho} = \sum_{(\mu\nu\rho)} T^\kappa_{\nu\mu}R_{\beta\alpha\kappa\rho}. \tag{43}$$

In a *GRT* model it follows that

$$\begin{gathered}\left(\mathring{R}^\star_{\mu\rho\lambda\sigma} + \mathring{R}^\star_{\mu\lambda\sigma\rho} + \mathring{R}^\star_{\mu\sigma\rho\lambda} + \frac{1}{2}\mathring{R}\epsilon_{\mu\rho\lambda\sigma}\right) = \epsilon_{\rho\lambda\sigma\kappa}\mathring{T}_\mu^{\ \kappa} \Leftrightarrow G_{\mu\nu} = T_{\mu\nu}, \\ \sum_{(\mu\alpha\beta)} \mathring{R}_{\mu\rho\alpha\beta} = 0, \qquad \sum_{(\mu\nu\rho)} \mathring{D}_\mu \mathring{R}_{\beta\alpha\nu\rho} = 0.\end{gathered} \tag{44}$$

Remark 10. Before proceeding we want to emphasize that *equations (20) and (21)* (the Bianchi identities) do not imply in general in the validity of the analogous equations for the duals of the torsion and curvature 2-forms, i.e. in general[13]

$$\mathbf{D}\underset{g}{\star}\mathcal{T}^\alpha \neq \underset{g}{\star}\mathcal{R}^\alpha_\beta \wedge \theta^\beta, \tag{45}$$

$$\mathbf{D}\underset{g}{\star}\mathcal{R}^\alpha_\beta \neq 0. \tag{46}$$

5. Are $\underset{g}{\star}\mathcal{T}^\alpha$ and $\underset{g}{\star}\mathcal{R}^\alpha_\beta$ the torsion and curvature 2-forms of any connection?

Despite the fact aforementioned in the last remark, we may pose the question: can $\underset{g}{\star}\mathcal{T}^\alpha$ and $\underset{g}{\star}\mathcal{R}^\alpha_\beta$ be the torsion and curvature 2-forms of a $\boldsymbol{g}$-metric compatible connection, say D', which defines on M the Riemann–Cartan structure $(M, \boldsymbol{g}, D')$ where also an Einstein-like equation is valid? If the answer is positive, the following set of equations must hold:

$$\begin{aligned}&\underset{g}{\star}\mathcal{G}'_\mu = \underset{g}{\star}\mathcal{R}'_\mu - \tfrac{1}{2}R'\underset{g}{\star}\theta_\mu = \underset{g}{\star}\mathbf{T}'_\mu \\ &\mathbf{D}'\mathcal{T}'^\alpha = \mathcal{R}'^\alpha_\beta \wedge \theta^\beta, \\ &\mathbf{D}'\mathcal{R}'^\alpha_\beta = 0,\end{aligned} \tag{47}$$

[13] In particular, a correct expression for $\mathbf{D}\underset{g}{\star}\mathcal{T}^\alpha$ has been found in [28].

and since by hypothesis $\underset{g}{\star}\mathcal{T}^\alpha = \mathcal{T}'^\alpha$ and $\underset{g}{\star}\mathcal{R}^\alpha_\beta = \mathcal{R}'^\alpha_\beta$, considering $\mathcal{G}^*_\mu = \mathcal{G}'_\mu$, $\mathcal{R}^*_\mu = \mathcal{R}'_\mu$, $R^* = R' = g^{\mu\beta} R^{*\rho}_{\mu\ \rho\beta}$ it must be

$$\begin{aligned}
&\underset{g}{\star}\mathcal{G}^*_\mu = \underset{g}{\star}\mathcal{R}^*_\mu - \tfrac{1}{2}R^* \underset{g}{\star}\theta_\mu = \underset{g}{\star}\mathbf{T}'_\mu \\
&\mathbf{D}' \underset{g}{\star}\mathcal{T}^\alpha = \underset{g}{\star}\mathcal{R}^\alpha_\beta \wedge \theta^\beta, \\
&\mathbf{D}' \underset{g}{\star}\mathcal{R}^\alpha_\beta = 0,
\end{aligned} \tag{48}$$

or in component form (and obvious notation)

$$\left(R_{\mu\rho\lambda\sigma} + R_{\mu\lambda\sigma\rho} + R_{\mu\sigma\rho\lambda} + \frac{1}{2}R^*\epsilon_{\mu\rho\lambda\sigma}\right) = \epsilon_{\rho\lambda\sigma\kappa}T'^{\ \kappa}_{\mu} \Longleftrightarrow G'_{\mu\nu} = T'_{\mu\nu}, \tag{49}$$

$$\sum_{(\mu\alpha\beta)} R^*_{\mu\rho\alpha\beta} = \sum_{(\mu\alpha\beta)} \left(D'_\mu T^*_{\rho\alpha\beta} - T^{*\kappa}_{\mu\beta} T^*_{\rho\kappa\alpha}\right), \tag{50}$$

$$\sum_{(\mu\nu\rho)} D'_\mu R^*_{\beta\alpha\nu\rho} = \sum_{(\mu\nu\rho)} T^{*\kappa}_{\nu\mu} R^*_{\beta\alpha\kappa\rho}. \tag{51}$$

Consequently, among the possible constraints in order to have a positive answer concerning the question in the head of the section, the following two non-trivial constraints are derived.

(a) Using equations (41) and (50), it follows that

$$\epsilon_{\rho\lambda\sigma\kappa}T^\kappa_\mu - \frac{1}{2}R\epsilon_{\mu\rho\lambda\sigma} = \sum_{(\mu\alpha\beta)} \left(D'_\mu T^*_{\rho\alpha\beta} - T^{*\kappa}_{\mu\beta} T^*_{\rho\kappa\alpha}\right). \tag{52}$$

(b) Using equations (49) (42) we must have

$$\epsilon_{\rho\lambda\sigma\kappa}T'^\kappa_\mu - \frac{1}{2}R^*\epsilon_{\mu\rho\lambda\sigma} = \sum_{(\mu\alpha\beta)} \left(D_\mu T_{\rho\alpha\beta} - T^\kappa_{\mu\beta} T_{\rho\kappa\alpha}\right). \tag{53}$$

Let us analyze what those constraints imply if we start with $(M, \boldsymbol{g}, \mathring{D})$, a Lorentzian structure (part of a Lorentzian spacetime structure) representing a gravitational field in *GRT*. In this case the second member of equation (53) must equal zero, and taking into account that $\mathring{R} = \mathring{T}' := \mathring{T}^\kappa_\kappa$ and $\mathring{R}^* = -\mathring{T}' := \mathring{T}'^\kappa_\kappa$ we obtain that the structure $(M, \boldsymbol{g}, \mathring{D}')$ must also be *torsion-free* and the following constraints must hold:

$$\begin{aligned}
&\epsilon_{\rho\lambda\sigma\kappa}\mathring{T}^\kappa_\mu = -\frac{1}{2}\mathring{T}\epsilon_{\mu\rho\lambda\sigma}, \qquad \epsilon_{\rho\lambda\sigma\kappa}\mathring{T}'^\kappa_\mu = -\frac{1}{2}\mathring{T}'\epsilon_{\mu\rho\lambda\sigma}, \\
&\sum_{(\mu\nu\rho)} \mathring{D}'_\mu R^*_{\beta\alpha\nu\rho} = \sum_{(\mu\nu\rho)} \mathring{D}_\mu R_{\beta\alpha\nu\rho}.
\end{aligned} \tag{54}$$

6. A particular case

Suppose we have as postulated[14] in [1] a Riemann–Cartan structure where equations (41)–(43) read

[14] It is obvious from our previous considerations that the equation $(R^\star_{\mu\rho\lambda\sigma} + R^\star_{\mu\lambda\sigma\rho} + R^\star_{\mu\sigma\rho\lambda}) = \epsilon_{\rho\lambda\sigma\kappa}T_\mu^{\ \kappa}$ presented in [1] as an identity is in general *wrong* and invalidates most of the conclusions of that paper. Also note that in [1] it is defined a Hodge dual with respect to the first pair of indices. However, since they start from a Lorentzian structure (where torsion is null) we have the validity of equation (9) and so in deriving equation (41) taking the dual with respect to the first or second pair of indices does not matter.

$$\left(R^{\star}_{\mu\rho\lambda\sigma} + R^{\star}_{\mu\lambda\sigma\rho} + R^{\star}_{\mu\sigma\rho\lambda}\right) = \epsilon_{\rho\lambda\sigma\kappa} T_{\mu}^{\ \kappa} \Longleftrightarrow G_{\mu\nu} = T_{\mu\nu}, \tag{55}$$

$$\sum_{(\mu\alpha\beta)} R_{\mu\rho\alpha\beta} = \epsilon_{\rho\alpha\beta\kappa} \Theta^{\kappa}_{\mu}, \tag{56}$$

$$\sum_{(\mu\nu\rho)} D_{\mu} R_{\beta\alpha\nu\rho} = 0. \tag{57}$$

It is obvious that we must then have

$$R = 0, \qquad \epsilon_{\rho\alpha\beta\kappa} \Theta^{\kappa}_{\mu} = \sum_{(\mu\alpha\beta)} \left(D_{\mu} T_{\rho\alpha\beta} - T^{\kappa}_{\mu\beta} T_{\rho\kappa\alpha}\right), \qquad \sum_{(\mu\nu\rho)} T^{\kappa}_{\nu\mu} R_{\beta\alpha\kappa\rho} = 0,$$

and comparing equation (53) with equation (47) we obtain

$$\epsilon_{\rho\lambda\sigma\kappa} T'^{\kappa}_{\mu} + \tfrac{1}{2} T'^{\kappa}_{\kappa} \epsilon_{\mu\rho\lambda\sigma} = \epsilon_{\rho\alpha\beta\kappa} \Theta^{\kappa}_{\mu}. \tag{58}$$

So a Riemann–Cartan structure satisfying equations (55)–(57) is possible only for matter distributions with $T = T^{\kappa}_{\kappa} = 0$ and which obey very stringent constraints.

Also, [1] choose as 'dual equations' the following set:

$$\left(R_{\mu\rho\lambda\sigma} + R_{\mu\lambda\sigma\rho} + R_{\mu\sigma\rho\lambda} + \frac{1}{2} R^{*} \epsilon_{\mu\rho\lambda\sigma}\right) = \epsilon_{\rho\lambda\sigma\kappa} \Theta^{\kappa}_{\mu} \Longleftrightarrow G^{*}_{\mu\nu} = \Theta_{\mu\nu}, \tag{59}$$

$$\sum_{(\mu\alpha\beta)} R^{*}_{\mu\rho\alpha\beta} = \epsilon_{\rho\alpha\beta\kappa} T^{\kappa}_{\mu}, \tag{60}$$

$$\sum_{(\mu\nu\rho)} D'_{\mu} R^{*}_{\beta\alpha\nu\rho} = 0 \tag{61}$$

which, of course, must imply

$$R^{*} = 0, \tag{62}$$

$$\epsilon_{\rho\alpha\beta\kappa} T^{\kappa}_{\mu} = \sum_{(\mu\alpha\beta)} \left(D'_{\mu} T^{*}_{\rho\alpha\beta} - T^{*\kappa}_{\mu\beta} T^{*}_{\rho\kappa\alpha}\right), \tag{63}$$

$$\sum_{(\mu\nu\rho)} T^{*\kappa}_{\nu\mu} R^{*}_{\beta\alpha\kappa\rho} = 0. \tag{64}$$

Comparing equation (64) with equation (52) implies again that $R = 0$. So we end with the following constraints, necessary for the validity of the equations proposed in [1]:

$$T'^{\kappa}_{\mu} = \Theta^{\kappa}_{\mu}, \qquad T = T^{\kappa}_{\kappa} = 0, \qquad \Theta = \Theta^{\kappa}_{\kappa} = 0,$$

$$\epsilon_{\rho\alpha\beta\kappa} \Theta^{\kappa}_{\mu} = \sum_{(\mu\alpha\beta)} \left(D_{\mu} T_{\rho\alpha\beta} - T^{\kappa}_{\mu\beta} T_{\rho\kappa\alpha}\right), \qquad \epsilon_{\rho\alpha\beta\kappa} T^{\kappa}_{\mu} = \sum_{(\mu\alpha\beta)} \left(D'_{\mu} T^{*}_{\rho\alpha\beta} - T^{*\kappa}_{\mu\beta} T^{*}_{\rho\kappa\alpha}\right), \tag{65}$$

$$\sum_{(\mu\nu\rho)} T^{\kappa}_{\nu\mu} R_{\beta\alpha\kappa\rho} = 0, \qquad \sum_{(\mu\nu\rho)} T^{*\kappa}_{\nu\mu} R^{*}_{\beta\alpha\kappa\rho} = 0.$$

Such constraints are clearly violated by the examples in [1].

7. Is there a metric $\boldsymbol{g}'$ and a metric connection D' such that $\underset{\boldsymbol{g}'}{\star}\mathcal{T}^{\alpha}$ and $\underset{\boldsymbol{g}'}{\star}\mathcal{R}^{\alpha}_{\beta}$ are their torsion and curvature forms?

Now, we can also put the question: in which conditions may we conceive that $\underset{\boldsymbol{g}}{\star}\mathcal{T}^{\alpha}$ and $\underset{\boldsymbol{g}'}{\star}\mathcal{R}^{\alpha}_{\beta}$ are the torsion and curvature 2-forms of a $\boldsymbol{g}'$-metric compatible connection, say D', which defines on M the Riemann–Cartan structure $(M, \boldsymbol{g}', D')$ where an Einstein-like equation holds, i.e. (with obvious notation) the validity of the following set of equations $\left(R' = g'^{\mu\beta} R'^{\rho}_{\mu\rho\beta}\right)$:

$$\begin{aligned}
&\underset{\boldsymbol{g}'}{\star}\mathcal{G}'_\mu = \underset{\boldsymbol{g}'}{\star}\mathcal{R}'_\mu - \tfrac{1}{2}R' \underset{\boldsymbol{g}'}{\star}\theta_\mu = \underset{\boldsymbol{g}'}{\star}\mathbf{T}'_\mu \\
&\mathbf{D}'\mathcal{T}'^\alpha = \mathcal{R}'^\alpha_\beta \wedge \theta^\beta, \\
&\mathbf{D}'\mathcal{R}'^\alpha_\beta = 0.
\end{aligned} \tag{66}$$

Since by hypothesis we must have $\underset{\boldsymbol{g}}{\star}\mathcal{T}^\alpha = \mathcal{T}'^\alpha$ and $\underset{\boldsymbol{g}}{\star}\mathcal{R}^\alpha_\beta = \mathcal{R}'^\alpha_\beta$, calling $R^* = g'^{\mu\beta} R^{*\rho}_{\mu\rho\beta}$ the set of equations (66) must be equal to

$$\begin{aligned}
&\underset{\boldsymbol{g}'}{\star}\mathcal{G}^*_\mu = \underset{\boldsymbol{g}'}{\star}\mathcal{R}^*_\mu - \tfrac{1}{2}R^* \underset{\boldsymbol{g}'}{\star}\theta_\mu = \underset{\boldsymbol{g}'}{\star}\mathbf{T}'_\mu \\
&\mathbf{D}' \underset{\boldsymbol{g}}{\star}\mathcal{T}'^\alpha = \underset{\boldsymbol{g}}{\star}\mathcal{R}^\alpha_\beta \wedge \theta^\beta, \\
&\mathbf{D}' \underset{\boldsymbol{g}}{\star}\mathcal{R}'^\alpha_\beta = 0,
\end{aligned} \tag{67}$$

which are similar but not identical to the set given by equation (48). Due to their complexity we shall not inspect the nature of those equations solutions, a problem postponed for another publication.

Remark 11. The constraints concerned in this case are more involved than in the previous case, but we want to emphasize here that if we start with $(M, \boldsymbol{g}, \mathring{D})$, a Lorentzian structure (part of a Lorentzian spacetime structure) representing a gravitational field in *GRT*, the structure $(M, \boldsymbol{g}', \mathring{D}')$ will be also torsion-free. Here we recall that [10] investigated long ago a similar problem (but only in the linear approximation) and found a positive answer for the question at the head of this section.

8. Conclusions

In this paper we present the *correct* constraints that must be satisfied by any theory (in a four-dimensional manifold) that intends to provide a dual presentation of the gravitational field equations for a general Riemann–Cartan theory. We compare our results with some of those proposed by authors quoted in the introduction and present some constructive criticisms. We hope that since the subject of duality becomes more important each day in, e.g., non-Abelian gauge theories, gravity and M-theory our results shall become appreciated.

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Appendix A. Exterior covariant derivative D

Sometimes equations (14) are written by some authors as

$$\mathbf{D}\theta^\rho = \mathcal{T}^\rho, \qquad \text{“}\mathbf{D}\omega^\rho_\mu = \mathcal{R}^\rho_\mu\text{”}.$$

and $\mathbf{D} : \sec \Lambda T^*M \to \sec \Lambda T^*M$ is said to be the *exterior covariant derivative* related to the connection D. The second of equations (14) has been printed with quotation marks due to the fact that it is not a correct equation. Indeed, a *legitimate* exterior covariant derivative operator[15] is a concept that can be defined for $(p+q)$-indexed r-form fields[16] as follows. Suppose that $X \in \sec T_p^{r+q}M$ and let $X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \in \sec \Lambda^r T^*M$, such that for $v_i \in \sec TM$, $i = 0, 1, 2, \ldots, r$, then $X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(v_1, \ldots, v_r) = X\big(v_1, \ldots, v_r, e_{\nu_1}, \ldots, e_{\nu_q}, \theta^{\mu_1}, \ldots, \theta^{\mu_p}\big)$. The exterior covariant differential $\mathbf{D}$ of $X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ on a manifold with a general connection D is the mapping

$$\mathbf{D}\colon\quad \sec \Lambda^r T^*M \to \sec \Lambda^{r+1} T^*M, \qquad 0 \leqslant r \leqslant 4, \tag{A.1}$$

such that[17]

$$\begin{aligned}(r+1)&\mathbf{D}X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(v_0, v_1, \ldots, v_r)\\ &= \sum_{\nu=0}^{r} (-1)^{\nu} D_{\mathbf{e}_\nu} X(v_0, v_1, \ldots, \check{v}_\nu, \ldots, v_r, \boldsymbol{e}_{\nu_1}, \ldots, \boldsymbol{e}_{\nu_q}, \theta^{\mu_1}, \ldots, \theta^{\mu_p})\\ &\quad - \sum_{0\leqslant\lambda,\varsigma\leqslant r} (-1)^{\nu+\varsigma} X(\mathbf{T}(v_\lambda, v_\varsigma), v_0, v_1, \ldots, \check{v}_\lambda, \ldots, \check{v}_\varsigma, \ldots, v_r, \boldsymbol{e}_{\nu_1}, \ldots, \boldsymbol{e}_{\nu_q}, \theta^{\mu_1}, \ldots, \theta^{\mu_p}).\end{aligned} \tag{A.2}$$

Then, we may verify that

$$\begin{aligned}\mathbf{D}X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} &= \mathrm{d}X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} + \omega^{\mu_1}_{\mu_s} \wedge X^{\mu_s\ldots\mu_p}_{\nu_1\ldots\nu_q} + \cdots + \omega^{\mu_1}_{\mu_s} \wedge X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}\\ &\quad - \omega^{\nu_s}_{\nu_1} \wedge X^{\mu_1\ldots\mu_p}_{\nu_s\ldots\nu_q} - \cdots - \omega^{\mu_1}_{\mu_s} \wedge X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_s}.\end{aligned}$$

Remark 12. Note that if equation (A.2) is applied on any one of the connection 1-forms ω^{μ}_{ν} we would obtain $\mathbf{D}\omega^{\mu}_{\nu} = \mathrm{d}\omega^{\mu}_{\nu} + \omega^{\mu}_{\alpha} \wedge \omega^{\alpha}_{\nu} - \omega^{\alpha}_{\nu} \wedge \omega^{\mu}_{\alpha}$. So we see that the symbol $\mathbf{D}\omega^{\mu}_{\nu}$ in equation (14), supposedly defining the curvature 2-forms, is simply wrong, despite this being an equation printed in many physics textbooks and many professional articles.

A.1. Properties of D

The exterior covariant derivative $\mathbf{D}$ satisfy the following properties:

(a) For any $X^J \in \sec \Lambda^r T^*M$ and $Y^K \in \sec \Lambda^s T^*M$ are sets of indexed forms[18],

$$\mathbf{D}(X^J \wedge Y^K) = \mathbf{D}X^J \wedge Y^K + (-1)^{rs} X^J \wedge \mathbf{D}Y^K. \tag{A.3}$$

(b) For any $X^{\mu_1\ldots\mu_p} \in \sec \Lambda^r T^*M$,

$$\mathbf{DD}X^{\mu_1\ldots\mu_p} = \mathrm{d}X^{\mu_1\ldots\mu_p} + \mathcal{R}^{\mu_1}_{\mu_s} \wedge X^{\mu_s\ldots\mu_p} + \cdots + \mathcal{R}^{\mu_p}_{\mu_s} \wedge X^{\mu_1\ldots\mu_s}. \tag{A.4}$$

(c) For any metric-compatible connection D if $g = g_{\mu\nu}\theta^\mu \otimes \theta^\nu$ then $\mathbf{D}g_{\mu\nu} = 0$.

Appendix B. Relation between the Riemann curvature tensors of the Lévi-Cività connection of $\mathring{\boldsymbol{g}}$ and a $\boldsymbol{g}$-compatible Riemann–Cartan connection

Let $(M, \mathring{\boldsymbol{g}}, \mathring{D})$ and $(M, \boldsymbol{g}, D)$ be respectively a Lorentzian and a Riemann–Cartan structure[19] on the same manifold M such that

$$\mathring{D}\mathring{\boldsymbol{g}} = 0, \qquad D\boldsymbol{g} = 0, \tag{B.1}$$

[15] Sometimes also called the exterior covariant differential.

[16] Which is not the case of the connection 1-forms ω^{α}_{β}, despite the name. More precisely, the ω^{α}_{β} are not true indexed forms, i.e. there does not exist a tensor field $\boldsymbol{\omega}$ such that $\boldsymbol{\omega}(e_i, e_\beta, \vartheta^\alpha) = \omega^{\alpha}_{\beta}(e_i)$.

[17] As usual the inverted hat over a symbol (in equation (A.2)) means that the corresponding symbol is missing in the expression.

[18] Multi indices are here represented by J and K.

[19] Note that $(M, \boldsymbol{g}, D)$ and $(M, \mathring{\boldsymbol{g}}, \mathring{D})$ are in general *Riemann–Cartan–Weyl* structures. More general formulas relating two arbitrary general connections may be found, e.g., in [27].

with the *nonmetricity* of D associated with $\mathring{g}$ being given by $\mathbf{Q} := -D\mathring{g}$. Let moreover the connection coefficients of $\mathring{D}$ and D in the arbitrary bases dual bases $\{e_\alpha\}$ and $\{\theta^\rho\}$ for $TU \subset TM$ and $T^*U \subset T^*M$ be

$$\mathring{D}_{\partial_\alpha}\theta^\rho = -\mathring{\Gamma}^\rho_{\alpha\beta}\theta^\beta, \qquad D_{\partial_\alpha}\theta^\rho = -L^\rho_{\alpha\beta}\theta^\beta, \tag{B.2}$$

and $Q_{\alpha\beta\sigma} = -D_\alpha\mathring{g}_{\beta\sigma}$. Define the components of the *strain tensor* of the connection D (associated with $\mathring{D}$) by

$$S^\rho_{\alpha\beta} = \left(L^\rho_{\alpha\beta} + L^\rho_{\alpha\beta}\right) - \left(\mathring{\Gamma}^\rho_{\alpha\beta} + \mathring{\Gamma}^\rho_{\alpha\beta}\right) \tag{B.3}$$

It is trivially established that

$$L^\rho_{\alpha\beta} = \mathring{\Gamma}^\rho_{\alpha\beta} + \tfrac{1}{2}T^\rho_{\alpha\beta} + \tfrac{1}{2}S^\rho_{\alpha\beta}. \tag{B.4}$$

where $\mathring{\Gamma}^\rho_{\alpha\beta}$ are the components of the Lévi-Cività connection of $\boldsymbol{g}$ and $T^\rho_{\alpha\beta}$ are the components of the torsion tensor of D[20].

Equation (B.4) can be used to relate the covariant derivatives with respect to the connections $\mathring{D}$ and D of any tensor field on the manifold. In particular, recalling that $\mathring{D}_\alpha\mathring{g}_{\beta\sigma} = e_\alpha(\mathring{g}_{\beta\sigma}) - \mathring{g}_{\mu\sigma}\mathring{\Gamma}^\mu_{\alpha\beta} - \mathring{g}_{\beta\mu}\mathring{\Gamma}^\mu_{\alpha\sigma} = 0$, we obtain the expression of the nonmetricity tensor of D in terms of the torsion and the strain, namely

$$Q_{\alpha\beta\sigma} = \tfrac{1}{2}\left(\mathring{g}_{\mu\sigma}T^\mu_{\alpha\beta} + \mathring{g}_{\beta\mu}T^\mu_{\alpha\sigma}\right) + \tfrac{1}{2}\left(\mathring{g}_{\mu\sigma}S^\mu_{\alpha\beta} + \mathring{g}_{\beta\mu}S^\mu_{\alpha\sigma}\right). \tag{B.5}$$

Equation (B.5) can be inverted to yield the expression of the strain in terms of the torsion and the nonmetricity. We obtain

$$S^\rho_{\alpha\beta} = \mathring{g}^{\rho\sigma}(Q_{\alpha\beta\sigma} + Q_{\beta\sigma\alpha} - Q_{\sigma\alpha\beta}) - \mathring{g}^{\rho\sigma}\left(\mathring{g}_{\beta\mu}T^\mu_{\alpha\sigma} + \mathring{g}_{\alpha\mu}T^\mu_{\beta\sigma}\right). \tag{B.6}$$

From equations (B.5) and (B.6) it is clear that nonmetricity and strain can be used interchangeably in the description of the geometry of a Riemann–Cartan–Weyl space. In particular, we have the relation

$$Q_{\alpha\beta\sigma} + Q_{\sigma\alpha\beta} + Q_{\beta\sigma\alpha} = S_{\alpha\beta\sigma} + S_{\sigma\alpha\beta} + S_{\beta\sigma\alpha}, \qquad \text{where} \quad S_{\alpha\beta\sigma} = \mathring{g}_{\rho\sigma}S^\rho_{\alpha\beta}. \tag{B.7}$$

In order to simplify our next equations, let us introduce the notation

$$K^\rho_{\alpha\beta} = L^\rho_{\alpha\beta} - \mathring{\Gamma}^\rho_{\alpha\beta} = \tfrac{1}{2}\left(T^\rho_{\alpha\beta} + S^\rho_{\alpha\beta}\right). \tag{B.8}$$

From equation (B.6) it follows that

$$K^\rho_{\alpha\beta} = -\tfrac{1}{2}\mathring{g}^{\rho\sigma}(D_\alpha\mathring{g}_{\beta\sigma} + D_\beta\mathring{g}_{\sigma\alpha} - D_\sigma\mathring{g}_{\alpha\beta}) - \tfrac{1}{2}\mathring{g}^{\rho\sigma}\left(\mathring{g}_{\mu\alpha}T^\mu_{\sigma\beta} + \mathring{g}_{\mu\beta}T^\mu_{\sigma\alpha} - \mathring{g}_{\mu\sigma}T^\mu_{\alpha\beta}\right). \tag{B.9}$$

Note also that for $D\mathring{g} = 0$, $K^\rho_{\alpha\beta}$ is the so-called *contorsion tensor*.

Returning to equation (B.4), we obtain now the relation between the curvature tensor $R_\mu{}^\rho{}_{\alpha\beta}$ associated with the connection D and the Riemann curvature tensor $\mathring{R}_\mu{}^\rho{}_{\alpha\beta}$ of the Lévi-Cività connection D associated with the metric $\boldsymbol{g}$. We obtain, by a straightforward calculation,

$$R_\mu{}^\rho{}_{\alpha\beta} = \mathring{R}_\mu{}^\rho{}_{\alpha\beta} + J_\mu{}^\rho{}_{[\alpha\beta]}, \tag{B.10}$$

where

$$J_\mu{}^\rho{}_{\alpha\beta} = \mathring{D}_\alpha K^\rho_{\beta\mu} - K^\rho_{\beta\sigma}K^\sigma_{\alpha\mu} = D_\alpha K^\rho_{\beta\mu} - K^\rho_{\alpha\sigma}K^\sigma_{\beta\mu} + K^\sigma_{\alpha\beta}K^\rho_{\sigma\mu}. \tag{B.11}$$

Multiplying both sides of equation (B.10) by $\frac{1}{2}\theta^\alpha \wedge \theta^\beta$ we obtain

$$\mathcal{R}^\rho_\mu = \mathring{\mathcal{R}}^\rho_\mu + \mathfrak{J}^\rho_\mu, \qquad \text{where} \quad \mathfrak{J}^\rho_\mu = \tfrac{1}{2}J_\mu{}^\rho{}_{[\alpha\beta]}\theta^\alpha \wedge \theta^\beta. \tag{B.12}$$

From equation (B.10) we also obtain the relation between the Ricci tensors of the connections D and $\mathring{D}$. The *Ricci tensor* is defined by

$$Ricci = R_{\mu\alpha}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \qquad \text{where} \quad R_{\mu\alpha} := R_\mu{}^\rho{}_{\alpha\rho}. \tag{B.13}$$

[20] More details may be found, e.g., in [27].

Then we have

$$R_{\mu\alpha} = \mathring{R}_{\mu\alpha} + J_{\mu\alpha}, \tag{B.14}$$

with

$$\begin{aligned} J_{\mu\alpha} &= \mathring{D}_\alpha K^\rho_{\rho\mu} - \mathring{D}_\rho K^\rho_{\alpha\mu} + K^\rho_{\alpha\sigma} K^\sigma_{\rho\mu} - K^\rho_{\rho\sigma} K^\sigma_{\alpha\mu} \\ &= D_\alpha K^\rho_{\rho\mu} - D_\rho K^\rho_{\alpha\mu} - K^\rho_{\sigma\alpha} K^\sigma_{\rho\mu} + K^\rho_{\rho\sigma} K^\sigma_{\alpha\mu}. \end{aligned} \tag{B.15}$$

Observe that since the connection D is arbitrary, its Ricci tensor will be *not* be symmetric in general. Then, since the Ricci tensor $\mathring{R}_{\mu\alpha}$ of $\mathring{D}$ is necessarily symmetric, we can split equation (B.14) into

$$R_{[\mu\alpha]} = J_{[\mu\alpha]}, \qquad R_{(\mu\alpha)} = \mathring{R}_{(\mu\alpha)} + J_{(\mu\alpha)}. \tag{B.16}$$

Appendix C. Some important identities

Let $(M, \boldsymbol{g})$ be a manifold and a Lorentzian metric as defined in section 1. Let moreover $\Lambda^p T^*M$ $(p = 0, 1, 2, 3, 4)$ be the bundle of homogeneous p-form fields and $\Lambda T^*M = \oplus_{p=0}^4 \Lambda^p T^*M$ the bundle of non homogeneous forms fields. We define in T^*M a metric field $\mathtt{g} \in \sec T_0^2 M$ such that concerning the general bases $\{\boldsymbol{e}_\mu\}$ and $\{\theta^\mu\}$ introduced in section 1, if $\boldsymbol{g} = g_{\mu\nu}\theta^\mu \otimes \theta^\nu$ and $\mathtt{g}= g^{\mu\nu}\boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu$ then $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$. In ΛT^*M we introduce a scalar product

$$\cdot : \Lambda T^*M \times \Lambda T^*M \to \Lambda T^*M \tag{C.1}$$

such that if $A, B \in \sec \Lambda^r T^*M$ are simple homogeneous r-forms with $A = u_1 \wedge \cdots \wedge u_r$ and $B = v_1 \wedge \cdots \wedge v_r$, $u_i, v_j \in \sec \Lambda^1 T^*M$, then $A \cdot B = \det(\mathtt{g}(u_i, v_j))$, where $(\mathtt{g}(u_i, v_j))$ means the matrix with entries $(\mathtt{g}(u_i, v_j))$. This scalar product is then extended by linearity and orthogonality to all ΛT^*M, and $A \cdot B = 0$ if $A \in \sec \Lambda^r T^*M$, and $B \in \sec \Lambda^s T^*M$ with $r \neq s$ and it is agreed that if $a, b \in \sec \Lambda^0 T^*M$, then $a \cdot b = ab$, the product of functions.

If the metric manifold $(M, \boldsymbol{g})$ is also endowed with an *orientation*, i.e. a volume n-vector $\tau_{\mathring{g}} \in \Lambda^4 T^*M$ such that $\tau_{\mathring{g}} \cdot \tau_{\mathring{g}} = -1$, then a natural isomorphism between sections of $\Lambda^r T^*M$ and $\Lambda^{4-r} T^*M$ $(r = 0, \ldots, 4)$ can be introduced. The *Hodge star operator* (or *Hodge dual*) is the linear mapping $\underset{\boldsymbol{g}}{\star} : \sec \Lambda^r T^*M \to \sec \Lambda^{4-r} T^*M$ such that

$$A \wedge \underset{\boldsymbol{g}}{\star} B = (A \cdot B)\tau_{\mathring{g}}, \tag{C.2}$$

for every $A, B \in \Lambda^r T^*M$. Of course, this operator is naturally extended to an isomorphism $\underset{\boldsymbol{g}}{\star} : \sec \Lambda T^*M \to \sec \Lambda T^*M$ by linearity. The inverse $\underset{\boldsymbol{g}}{\star}^{-1} : \sec \Lambda^r T^*M \to \sec \Lambda^{4-n} T^*M$ of the Hodge star operator is given by $\underset{\boldsymbol{g}}{\star}^{-1} = -(-1)^{r(4-r)} \underset{\boldsymbol{g}}{\star}$. For any$A, B \in \sec \Lambda T^*M$

$$A \cdot B = \langle \tilde{A} B \rangle_0 = \langle A \tilde{B} \rangle_0 = B \cdot A, \tag{C.3}$$

where $\tilde{A}$ means the the reverse of A. If $A = u_1 \wedge \cdots \wedge u_r$, then $\tilde{A} = u_r \wedge \cdots \wedge u_1$ and $\langle\,\rangle_0 : \sec \Lambda T^*M \to \sec \Lambda^0 T^*M$ is the projection of a general non homogeneous form into the $\Lambda^0 T^*M$ part.

Remark 13. Suppose that $\{\varepsilon_i\}$ is, e.g., an orthonormal basis of $\Lambda^1 T^*M$ and $\{\varepsilon^j\}$ is reciprocal basis, i.e. $\varepsilon_i \cdot \varepsilon^k = \delta^k_i$. Then any $Y \in \sec \Lambda^p T^*M$ can be written as

$$Y = \frac{1}{p!} Y^{j_1 \ldots j_p} \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p} = \frac{1}{p!} Y_{j_1 \cdots j_p} \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_p} \tag{C.4}$$

and

$$Y^{j_1 \ldots j_p} = Y \cdot (\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_p}), \qquad Y_{j_1 \ldots j_p} = Y \cdot (\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p}). \tag{C.5}$$

We define the right and left contractions of non homogeneous differential forms as follows. For arbitrary multiforms $X, Y, Z \in \sec \Lambda T^*M$, the left ($\underset{g}{\lrcorner}$) and right ($\underset{g}{\llcorner}$) contractions of X and Y are the mappings $\underset{g}{\lrcorner} : \sec \Lambda T^*M \times \sec \Lambda T^*M \to \sec \Lambda T^*M$ and $\underset{g}{\llcorner} : \sec \Lambda T^*M \times \sec \Lambda T^*M \to \sec \Lambda T^*M$ such that

$$(X \underset{g}{\lrcorner} Y) \cdot Z = Y \cdot (\tilde{X} \wedge Z), \qquad (X \underset{g}{\llcorner} Y) \cdot Z = X \cdot (Z \wedge \tilde{Y}). \tag{C.6}$$

These contracted products $\underset{g}{\lrcorner}$ and $\underset{g}{\llcorner}$ are inner derivations on ΛT^*M. Sometimes the contractions are called the interior products. Both contract products satisfy the left and right distributive laws but they are *not* associative.

Now some important properties of the contractions used in the calculations of the text are presented.

(i) For any $a, b \in \sec \Lambda^0 T^*M$, and $Y \in \sec \Lambda T^*M$

$$\begin{aligned} &a \underset{g}{\lrcorner} b = a \underset{g}{\llcorner} b = ab \text{ (product of functions)}, \\ &a \underset{g}{\lrcorner} Y = Y \underset{g}{\llcorner} a = aY \text{ (multiplication by scalars)}. \end{aligned} \tag{C.7}$$

(ii) If $a, b_1, \ldots, b_k \in \sec \Lambda T^*M$, then $a \underset{g}{\lrcorner} (b_1 \wedge \cdots \wedge b_k) = \sum_{j=1}^{k} (-1)^{j+1} (a \cdot b_j) b_1 \wedge \cdots \wedge \check{b}_j \wedge \cdots \wedge b_k$, where the symbol $\check{b}_j$ means that the b_j factor does not appear in the j-term of the sum.

(iii) For any $Y_j \in \sec \Lambda^j T^*M$ and $Y_k \in \sec \Lambda^k T^*M$ with $j \leqslant k$

$$Y_j \underset{g}{\lrcorner} Y_k = (-1)^{j(k-j)} Y_k \underset{g}{\llcorner} Y_j. \tag{C.8}$$

(iv) For any $Y_j \in \sec \Lambda^j T^*M$ and $Y_k \in \sec \Lambda^k T^*M$

$$Y_j \underset{g}{\lrcorner} Y_k = 0, \quad \text{if} \quad j > k, \qquad Y_j \underset{g}{\llcorner} Y_k = 0, \quad \text{if} \quad j < k. \tag{C.9}$$

(v) For any $X_k, Y_k \in \sec \Lambda^k T^*M$, then $X_k \underset{g}{\lrcorner} Y_k = Y_k \underset{g}{\llcorner} Y_k = \tilde{X}_k \cdot Y_k = X_k \cdot \tilde{Y}_k$.

(vi) For any $v \in \sec \Lambda^1 T^*M$ and $X, Y \in \sec \Lambda T^*M$, then $v \underset{g}{\lrcorner} (X \wedge Y) = (v \underset{g}{\lrcorner} X) \wedge Y + \hat{X} \wedge (v \underset{g}{\lrcorner} Y)$. Also, if $A, B \in \sec \Lambda^k T^*M$ then $A \underset{g}{\lrcorner} (B \underset{g}{\lrcorner} C) = (A \wedge B) \underset{g}{\lrcorner} C$, and $A \underset{g}{\llcorner} (B \underset{g}{\llcorner} C) = A \underset{g}{\llcorner} (B \wedge C)$.

(vii) if $A, B \in \sec \Lambda T^*M$ then

$$(A \underset{g}{\lrcorner} B) \cdot C = B \cdot (\tilde{A} \wedge C), \qquad (B_g \underset{g}{\llcorner} A) \cdot C = B \cdot (C \wedge \tilde{A}). \tag{C.10}$$

Finally we present some important identities involving contractions and the Hodge dual. Let $A_r \in \sec \Lambda^r T^*M$ and $B_s \in \sec \Lambda^s T^*M$, $r, s \geqslant 0$:

$$\begin{aligned} &A_r \wedge \underset{g}{\star} B_s = B_s \wedge \underset{g}{\star} A_r \quad r = s; \qquad A_r \cdot \underset{g}{\star} B_s = B_s \cdot \underset{g}{\star} A_r; \quad r + s = n, \\ &A_r \wedge \underset{g}{\star} B_s = (-1)^{r(s-1)} \underset{g}{\star} (\tilde{A}_r \underset{g}{\lrcorner} B_s); \quad r \leqslant s, \\ &A_r \underset{g}{\lrcorner} \underset{g}{\star} B_s = (-1)^{rs} \underset{g}{\star} (\tilde{A}_r \wedge B_s); \quad r + s \leqslant n, \\ &\underset{g}{\star} A_r = \tilde{A}_r \underset{g}{\lrcorner} \tau_{\mathring{g}}, \qquad \underset{g}{\star} \tau_{\mathring{g}} = -1, \qquad \underset{g}{\star} 1 = \tau_{\mathring{g}}. \end{aligned} \tag{C.11}$$

References

[1] Argurio R and Dehouck F 2009 Why not a di-NUT? Or gravitational duality and rotating solutions arXiv:0909.0542v1 [hep-th]
[2] Argurio R, Dehouck F and Houart L 2009 Supersymmetry and gravitational duality *Phys. Rev.* D **79** 125001 (arXiv:0810.4999v3 [hep-th])
[3] Barnich G and Troessaert C 2009 Manifest spin 2 duality with electric and magnetic sources *J. High Energy Phys.* JHEP01(2009)030 (arXiv:0812.0552v2 [hep-th])
[4] Benn I M and Tucker R W 1987 *An Introduction to Spinors and Geometry* (Bristol and New York: Adam Hilger)
[5] Bergshoeff E A, de Roo M, Kerstan S F, Kleinschmidt A and Riccioni F 2009 Dual gravity and matter *Gen. Rel. Grav* **41** 39–48 (arXiv:0803.1963v2 [hep-th])
[6] Bergshoeff E A, Hartong J, Hohm O, Hübscher M and Ortín T 2009 Gauge theories, duality relations and the tensor hierarchy JHEP04(2009)123 (arXiv:0901.2054v2 [hep-th])
[7] Bunster C, Cnockaert S, Henneaux M and Portugues R 2006 Monopoles for gravitation and for higher spin fields *Phys. Rev* D **73** 105014 (arXiv:hep-th/0601222v3)
[8] Choquet-Bruhat Y, DeWitt-Morette C and Dillard-Bleick M 1982 *Analysis, Manifolds and Physics* (revised edition) (Amsterdam: North Holland)
[9] Do Carmo M P 1982 *Riemannian Geometry* (Boston, MA: Birkhäuser)
[10] Dowker J S and Roche J A 1967 The gravitational analogues of magnetic monopoles *Proc. Phys. Soc.* **92** 1–8
[11] Fecko M 2006 *Differential Geometry and Lie Groups for Physicists* (Cambridge: Cambridge University Press)
[12] Fernández V V and Rodrigues W A Jr 2009 Gravitation as a plastic distortion of the Lorentz vacuum arXiv:0909.4472v2 [math-ph]
An enlarged version is to appear as book in the series Fernández V V and Rodrigues W A Jr 2010 *Fundamental Theories of Physics* (Heidelberg: Springer)
[13] Forder P W 1987 Gravitomagnetic poles and the quantization of frequency *Class. Quantum Grav.* **4** 703–10
[14] Garcia-Compean H, Nieto A, Obregon O and Ramirez C 1999 Dual description of supergravity MacDowell–Mansouri theory *Phys. Rev.* D **59** 124003 (arXiv:hep-th/9812175v2)
[15] Garcia-Compean H, Obregon O, Plebanski J F and Ramirez C 1998 Towards a gravitational analog to S-duality in non-Abelian gauge theories *Phys. Rev.* D **57** 7501–6 (arXiv:hep-th/9711115v3)
[16] Hehl F W, von der Heyde P, Kerlick G D and Nester J M 1976 General relativity with spin and torsion: foundations and prospects *Rev. Mod. Phys.* **48** 393–416
[17] Maia A Jr, Recami E, Rodrigues W A Jr and Rosa M A F 1990 Magnetic monopoles without string in the Kähler Clifford algebra bundle: a geometrical interpretation *J. Math. Phys.* **31** 502–5
[18] Maia A Jr and Rodrigues W A Jr 1994 A generalized Dirac's quantization condition for phenomenological non-Abelian magnetic monopoles *Mod. Phys. Lett* A **9** 81–8
[19] Mignani R 1978 Gravitational analogues of magnetic monopoles from the path-dependent formalism *Lett. Nuovo Cimento* **22** 597–600
[20] Mignani R 1980 Path-dependent approach to gravitational monopoles *Nuovo Cimento* **56** 201–9
[21] Nieto J A 1999 S-Duality for linearized gravity *Phys. Lett.* A **262** 274–81 (arXiv:hep-th/9910049v1)
[22] Nieto J A, Soccoro J and Obregon O 1996 Gauge theory of supergravity based on a self-dual spin connection *Phys. Rev. Lett.* **76** 3482–5
[23] Notte-Cuello E A, Rodrigues W A Jr and G Souza Q A 2007 The square of the dirac and spin-dirac operators on a Riemann–Cartan space(time) *Rep. Math. Phys.* **60** 135–57
[24] Quevedo F 1998 Duality and global symmetries *Nucl. Phys. Proc. Suppl.* A **61** 23–41 (arXiv:hep-th/9706210v1)
[25] Sachs R K and Wu H 1977 *General Relativity for Mathematicians* (New York: Springer)
[26] Rodrigues W A Jr and G Souza Q A 1993 The Clifford bundle and the nature of the gravitational field *Found. Phys.* **23** 1465–90
[27] Rodrigues W A Jr and Capelas de Oliveira E 2007 *The Many Faces of Maxwell, Dirac and Einstein Equations. A Clifford Bundle Approach* (Heidelberg: Springer)
[28] Rodrigues W A Jr 2007 Differential forms on Riemann (lorentzian) an Riemann–Cartan structures and some applications to physics *Ann. Fond. L. de Broglie* **32** 425–78 (special issue dedicated to torsion) (arXiv:0712.3067v6 [math-ph])
[29] Rosa M A F and Rodrigues W A Jr 1989 A geometrical theory of non topological magnetic monopoles *Mod. Phys. Lett.* A **4** 175 184
[30] Thirring W and Wallner R 1978 The use of exterior forms in Einstein's gravitational theory *Braz. J. Phys.* **8** 668–723
[31] Witten E 1995 On *S*-duality in Abelian Gauge theory *Selecta Math.* **1** 383–410 (arXiv:hep-th/9505186v1)